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Serret-Frenet Formulas for Octonionic Curves

Özcan Bektaş and Salim Yüce

ABSTRACT: In this paper, we define spatial octonionic curves (SOC) in $\mathbb{R}^7$ and octonionic curves (OC) in $\mathbb{R}^8$ by using octonions. Firstly, we determine Serret-Frenet equations, and curvatures of the SOC in $\mathbb{R}^7$. Then, Serret-Frenet equations for the OC in $\mathbb{R}^8$ are calculated with the help of Serret-Frenet equations of SOC in $\mathbb{R}^7$.

Key Words: Serret-Frenet formulas, Spatial octonionic curve, Octonionic curve, Curvature, Torsion.

Contents

1. Introduction

Octonions and quaternions are important subjects in differential geometry. Quaternionic curves play an important role in differential geometry. Spatial quaternion set (if the real part of quaternion is equal to zero, then the quaternion is called spatial quaternion) is isomorphic to Euclidean 3-space. Moreover, the set of real quaternions is isomorph to Euclidean 4-space. For that reason, Bharathi and Nagaraj studied the differential geometry of a smooth curve in Euclidean 4-space $\mathbb{R}^4$ [3]. The elements of $\mathbb{R}^4$ are identified with the quaternions. The Serret-Frenet apparatus for the quaternionic curves were determined by the Serret-Frenet apparatus for a main curve in $\mathbb{R}^3$ which is embedded in $\mathbb{R}^4$ [3]. Then, a lot of papers were published by using the quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$. For example, Karadag, Gunes and Sivridag determined the Serret-Frenet formulas for dual quaternion valued functions of a single real variable [11]. The quaternion valued functions, and quaternionic inclined curves were studied in the semi-Euclidean space by Coken and Tuna [6] and [15]. The curl in $\mathbb{R}^7$ introduced by the cross product in $\mathbb{R}^7$ by Peng and Yang [22].

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Octonions were defined independently by J. Thomas Graves and A. Cayley. The set of the octonions $\mathbb{O}$ are expressed as follows:

$$\mathbb{O} = \{\mathbb{A}_0 e_0 + \sum_{i=1}^{7} \mathbb{A}_i e_i;\ \mathbb{A}_i \in \mathbb{R}\},$$

where $e_i^2 = -1$, $e_0 e_i = e_i e_0 = e_i$, $(\forall i = 1, 2, ..., 7)$, $e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k$, δ_{ij} is Kronecker delta, ε_{ijk} is completely antisimetric tensor, $e_0 = +1$, and $(ijk) = (123), (145), (176), (246), (257), (347), (365)$ [5],[9],[14],[18]. There have been many investigations related to octonions. Researchers study octonions in analysis, physics, and differential geometry. Spatial octonion set (if the real part of octonion is equal to zero, then the real octonion is called the spatial octonion [16]. These octonions create the spatial octonion set and denoted by $\mathbb{O}_S$) is isomorph to $\mathbb{R}^7$, and the set of octonions is isomorphic to $\mathbb{R}^8$. In other words, $\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$ [13], [19]. The points in $\mathbb{R}^8$ can be represented by the octonions [1]. Thus, we can ask ourselves the following questions:

Can we define the octonionic curves in $\mathbb{R}^7$ and $\mathbb{R}^8$, and find the Serret-Frenet apparatus for the octonionic curve $\beta_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}$, $\beta_{\mathbb{O}}(s) = \sum_{i=0}^{7} \gamma_i(s) e_i$, $\beta_{\mathbb{O}}(s) = \gamma_0(s) e_0 + \gamma_{\mathbb{O}}(s)$ by using the Serret-Frenet apparatus for a main spatial octonionic curve, $\gamma_O : I \subset \mathbb{R} \to \mathbb{O}_S$, $\gamma_{\mathbb{O}}(s) = \sum_{i=1}^{7} \gamma_i(s) e_i$ in $\mathbb{R}^7$ which is embedded in $\mathbb{R}^8$? In this paper, we will answer the above questions. We transfer the concept of the quaternionic curve [3] in Euclidean 4-space to the concept of the octonionic curve in Euclidean 8-space by using the real octonions (We are dealing with real octonions in this paper and henceworth use just word octonions).

Our study is prepared as follows. In preliminaries part, we give fundamental definitions, properties, and informations about the octonion algebras. In section 3, we introduce the spatial octonionic curves (SOC) in $\mathbb{R}^7$, and the octonionic curves (OC) in $\mathbb{R}^8$. Then, we find the Serret-Frenet apparatus for SOC in $\mathbb{R}^7$. By using the Serret-Frenet apparatus for SOC in $\mathbb{R}^7$, we compute the Serret-Frenet apparatus for OC in $\mathbb{R}^8$.

2. Preliminaries

In this section, we denote the set of spatial octonions with $\mathbb{O}_S$, and the set of octonions with $\mathbb{O}$. Let us first give some fundamental notions of the octonions. The real octonion $\mathbb{A}$ is defined by

$$\mathbb{A} = \mathbb{A}_0 e_0 + \sum_{i=1}^{7} \mathbb{A}_i e_i,$$

where $\mathbb{A}_i$ are the real numbers components of the octonions, $e_i's$ $(i = 1, 2, \ldots, 7)$ are the unit octonions basis elements, and $e_0 = +1$ is the scalar element [5], [14],

[18]. The multiplication rules of the unit octonions basis elements are given by the following table

Table 1.The Multiplication Table of Unit Octonions Basis Elements

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The addition, the scalar multiplication, and the octonion multiplication are the operations of the set of the octonions. The sum of two octonions is defined by

$$\begin{aligned}\mathbb{A}\pm\mathbb{B} &= \sum_{i=0}^{7}\left(\mathbb{A}_i\pm\mathbb{B}_i\right)e_i\\ &= \left(\mathbb{A}_0e_0+\mathbb{A}_1e_1+\mathbb{A}_2e_2+\mathbb{A}_3e_3+\mathbb{A}_4e_4+\mathbb{A}_5e_5+\mathbb{A}_6e_6+\mathbb{A}_7e_7\right)\\ &\quad\pm\left(\mathbb{B}_0e_0+\mathbb{B}_1e_1+\mathbb{B}_2e_2+\mathbb{B}_3e_3+\mathbb{B}_4e_4+\mathbb{B}_5e_5+\mathbb{B}_6e_6+\mathbb{B}_7e_7\right).\end{aligned}$$

$\overline{\mathbb{A}}$ is called conjugate of the octonion $\mathbb{A}$, and conjugate of the octonion is defined by

$$\begin{aligned}\overline{\mathbb{A}} &= \mathbb{A}_0e_0-\mathbb{A}_1e_1-\mathbb{A}_2e_2-\mathbb{A}_3e_3-\mathbb{A}_4e_4-\mathbb{A}_5e_5-\mathbb{A}_6e_6-\mathbb{A}_7e_7\\ &= \mathbb{A}_0e_0-\sum_{i=1}^{7}\mathbb{A}_ie_i,\end{aligned}$$

where $\overline{e_0}=e_0$ and $\overline{e_j}=-e_j\ \ (j=1,\ldots,7)$ [8]. The octonion $\mathbb{A}$ has real part and vectorial part. So, the octonion $\mathbb{A}$ is decomposed with respect to its real ($S_{\mathbb{A}}$), and vectorial ($V_{\mathbb{A}}$) parts as follows:

$$S_{\mathbb{A}}=\frac{1}{2}\left(\mathbb{A}+\overline{\mathbb{A}}\right)=\mathbb{A}_0e_0,\ V_{\mathbb{A}}=\frac{1}{2}\left(\mathbb{A}-\overline{\mathbb{A}}\right)=\sum_{i=1}^{7}\mathbb{A}_ie_i[8],[13].$$

Let us denote octonions with a real number ($S_{\mathbb{A}}=\mathbb{A}_0$) in $\mathbb{R}$, and a vector $\left(V_{\mathbb{A}}=\sum_{i=1}^{7}\mathbb{A}_ie_i\right)$ in $\mathbb{R}^7$. The octonion is given by

$$\mathbb{A}=S_{\mathbb{A}}+V_{\mathbb{A}}.$$

The multiplication of two octonions is defined by

$$\mathbb{A}\times\mathbb{B}=S_{\mathbb{A}}S_{\mathbb{B}}-g\left(V_{\mathbb{A}},V_{\mathbb{B}}\right)+S_{\mathbb{A}}V_{\mathbb{B}}+S_{\mathbb{B}}V_{\mathbb{A}}+V_{\mathbb{A}}\wedge V_{\mathbb{B}}, \tag{2.1}$$

where $\forall \mathbb{A}, \mathbb{B} \in \mathbb{O}$, [12], [17], [20, 21]. We use the inner and the cross products in $\mathbb{R}^7$ in above equations [12]. The symmetric, non-degenerate real-valued bilinear form g is introduced by

$$g : \mathbb{O} \times \mathbb{O} \to \mathbb{R},\ g(\mathbb{A}, \mathbb{B}) = \frac{1}{2}\left(\mathbb{A} \times \overline{\mathbb{B}} + \mathbb{B} \times \overline{\mathbb{A}}\right),$$

where $\mathbb{A}, \mathbb{B} \in \mathbb{O}$. g is determined with the help of the real octonionic multiplication. g is called the octonionic inner product. Thus, $g(\mathbb{A}, \mathbb{B}) = \sum\limits_{i=0}^{7} \mathbb{A}_i \mathbb{B}_i$ [4], [7]. If $\mathbb{A} + \overline{\mathbb{A}} = 0$, then the octonion $\mathbb{A}$ is called the spatial octonion. The spatial octonion set is represented by

$$\mathbb{O}_S = \{\sum_{i=1}^{7} \mathbb{A}_i e_i;\ \mathbb{A}_i \in \mathbb{R}\},$$

where $e_i^2 = -1$, $e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k$, $(i, j, k = 1, 2, ..., 7)$, $(i \neq j \neq k, i \neq 0, j \neq 0, k \neq 0)$. The spatial octonion set is isomorphic to $\mathbb{R}^7$.

The vector product of two vectors is only defined in 3-dimensional Euclidean space, $\mathbb{R}^3$ and 7-dimensional Euclidean space, $\mathbb{R}^7$ [8], [12]. We express the vector product in $\mathbb{R}^7$ as follows. Let $\mathbb{A}$ and $\mathbb{B}$ be the spatial octonions. Then, the vector product in $\mathbb{R}^7$ is defined by

$$\mathbb{A} \wedge \mathbb{B} = \mathbb{A} \times \mathbb{B} + \langle \mathbb{A}, \mathbb{B} \rangle,$$

where $\mathbb{A} \times \mathbb{B} = (\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_5, \mathbb{A}_6, \mathbb{A}_7) \times (\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5, \mathbb{B}_6, \mathbb{B}_7)$ is defined by

$$\begin{aligned}
\mathbb{A} \times \mathbb{B} \quad = \quad & (\mathbb{A}_2\mathbb{B}_3 - \mathbb{A}_3\mathbb{B}_2 + \mathbb{A}_4\mathbb{B}_5 - \mathbb{A}_5\mathbb{B}_4 + \mathbb{A}_7\mathbb{B}_6 - \mathbb{A}_6\mathbb{B}_7, \\
& \mathbb{A}_3\mathbb{B}_1 - \mathbb{A}_1\mathbb{B}_3 + \mathbb{A}_4\mathbb{B}_6 - \mathbb{A}_6\mathbb{B}_4 + \mathbb{A}_5\mathbb{B}_7 - \mathbb{A}_7\mathbb{B}_5, \\
& \mathbb{A}_1\mathbb{B}_2 - \mathbb{A}_2\mathbb{B}_1 + \mathbb{A}_4\mathbb{B}_7 - \mathbb{A}_7\mathbb{B}_4 + \mathbb{A}_6\mathbb{B}_5 - \mathbb{A}_5\mathbb{B}_6, \\
& \mathbb{A}_5\mathbb{B}_1 - \mathbb{A}_1\mathbb{B}_5 + \mathbb{A}_6\mathbb{B}_2 - \mathbb{A}_2\mathbb{B}_6 + \mathbb{A}_7\mathbb{B}_3 - \mathbb{A}_3\mathbb{B}_7, \\
& \mathbb{A}_1\mathbb{B}_4 - \mathbb{A}_4\mathbb{B}_1 + \mathbb{A}_3\mathbb{B}_6 - \mathbb{A}_6\mathbb{B}_3 + \mathbb{A}_7\mathbb{B}_2 - \mathbb{A}_2\mathbb{B}_7, \\
& \mathbb{A}_1\mathbb{B}_7 - \mathbb{A}_7\mathbb{B}_1 + \mathbb{A}_2\mathbb{B}_4 - \mathbb{A}_4\mathbb{B}_2 + \mathbb{A}_5\mathbb{B}_3 - \mathbb{A}_3\mathbb{B}_5, \\
& \mathbb{A}_2\mathbb{B}_5 - \mathbb{A}_5\mathbb{B}_2 + \mathbb{A}_3\mathbb{B}_4 - \mathbb{A}_4\mathbb{B}_3 + \mathbb{A}_6\mathbb{B}_1 - \mathbb{A}_1\mathbb{B}_6),
\end{aligned}$$

and $\langle \mathbb{A}, \mathbb{B} \rangle = \sum\limits_{i=1}^{7} \mathbb{A}_i \mathbb{B}_i$ is standart inner product in $\mathbb{R}^7$ [5], [8]. Moreover, this vector product is given by [8], [14], [17] for all $\mathbb{A} = \sum\limits_{i=1}^{7} \mathbb{A}_i e_i = (\mathbb{A}_i)$, $1 \leq i \leq 7$ and $\mathbb{B} = \sum\limits_{i=1}^{7} \mathbb{B}_i e_i = (\mathbb{B}_i)$, $1 \leq i \leq 7$. The vector product in $\mathbb{R}^7$ satisfies the following properties:

i) **Distributive property:** $\mathbb{A} \wedge (\mathbb{B} + \mathbb{C}) = \mathbb{A} \wedge \mathbb{B} + \mathbb{A} \wedge \mathbb{C}$,

ii) **The vector product of the spatial octonion with itself is zero**, $\mathbb{A} \wedge \mathbb{A} = 0$,

iii) **Alternating property:** $\mathbb{A} \wedge \mathbb{B} = -\mathbb{B} \wedge \mathbb{A}$,
iv) $g(\mathbb{A}, \mathbb{A} \wedge \mathbb{B}) = 0$,
v) **The norm of the vector product of two spatial octonions:** $\|\mathbb{A} \wedge \mathbb{B}\| = \|\mathbb{A}\| \, \|\mathbb{B}\| \, sin\theta$,
vi)**Mixed scalar product:** $g(\mathbb{A} \wedge \mathbb{B}, \mathbb{C}) = g(\mathbb{B} \wedge \mathbb{C}, \mathbb{A}) = g(\mathbb{C} \wedge \mathbb{A}, \ \mathbb{B})$,
vii) $\mathbb{A} \wedge (\mathbb{A} \wedge \mathbb{B}) = g(\mathbb{A}, \mathbb{B}) \ \mathbb{A} - g(\mathbb{A}, \mathbb{A}) \, \mathbb{B}$.
The norm of the octonion $\mathbb{A}$ is denoted by

$$\|\mathbb{A}\| = \sqrt{\mathbb{A} \times \overline{\mathbb{A}}} = \sqrt{\sum_{i=0}^{7} \mathbb{A}_i^2}.$$

If $\|\mathbb{A}_0\| = 1$, then $\mathbb{A}_0$ is called the unit octonion [8], [12]. The inverse of an octonion is defined by [7]

$$\mathbb{A}^{-1} = \frac{\overline{\mathbb{A}}}{\|\mathbb{A}\|^2}, \ \ \mathbb{A} \neq 0.$$

If $\mathbb{A}$ and $\mathbb{B}$ octonions, then $(\mathbb{B} \times \mathbb{A}^{-1}) \times \mathbb{A} = \mathbb{B}$ or $\mathbb{A}^{-1} \times (\mathbb{A} \times \mathbb{B}) = \mathbb{B}$ [19].

In differential geometry, curve theory is a developed subject of study. The planar curve in 2-dimensional Euclidean space $\mathbb{R}^2$, the space curve in 3-dimensional Euclidean space $\mathbb{R}^3$, and the space curve in n-dimensional Euclidean space $\mathbb{R}^n$ were defined. Since $\mathbb{R}^2$ is corresponding to $\mathbb{C}$, then the planar curves in $\mathbb{R}^2$ were studied with respect to complex numbers. Accordingly, since $\mathbb{R}^3$ and $\mathbb{R}^4$ are corresponding to the spatial quaternion set $\mathbb{H}_S$ and the real quaternion set $\mathbb{H}$, respectively, then the space curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ were studied in terms of quaternions. The main question of our paper is: Can we study the space curves in $\mathbb{R}^7$ and $\mathbb{R}^8$ by using the octonions? To answer this question, we have to know strong evidence. Our main evidence is that $\mathbb{R}^7$ and $\mathbb{R}^8$ are corresponding to $\mathbb{O}_S$ and $\mathbb{O}$, respectively. In this paper, we use the Serret-Frenet formulas for the well known space curve in $\mathbb{R}^7$ and $\mathbb{R}^8$ (We know that space curves were defined in $\mathbb{R}^n$. So, if we take $n = 7, 8$, we get the $\mathbb{R}^7$ and $\mathbb{R}^8$).

The Serret Frenet frame and curvatures in $\mathbb{R}^8$: Let $\Gamma : I \subset \mathbb{R} \to \mathbb{R}^8$ be a unit speed space curve in $\mathbb{R}^8$ and $\{\mathbf{U}_j\}$, $1 \leq j \leq 8$ be the Serret Frenet 8-frame related to Γ. The Serret-Frenet formulas for the curve $\Gamma : I \subset \mathbb{R} \to \mathbb{R}^8$ are given by

$$\begin{aligned} &\mathbf{U}_1^{'}(s) = k_1(s)\,\mathbf{U_2}(s) \\ &\mathbf{U}_{\mathbf{m}}^{'}(s) = -k_{m-1}(s)\,\mathbf{U}_{m-1}(s) + k_m(s)\,\mathbf{U}_{m+\mathbf{1}}(s), \ 2 \leq m \leq 7 \\ &\mathbf{U}_{\mathbf{8}}^{'}(s) = -k_7(s)\,\mathbf{U_7}(s). \end{aligned} \tag{2.2}$$

On the other hand, $\mathbf{U}_j(s) = \frac{\mathbf{E}_j(s)}{\|\mathbf{E}_j(s)\|}$, $k_j(s) = \left\langle \mathbf{U}_j^{'}(s), \mathbf{U}_{j+1}(s) \right\rangle = \frac{\|\mathbf{E}_j+1(s)\|}{\|\mathbf{E}_j(s)\|}$ for $1 \leq j \leq 8$ [10].

In this paper, we compute the above Eq. (2.2) by the help of the octonions for SOC in $\mathbb{R}^7$. Then, by using the Serret-Frenet apparatus for SOC in $\mathbb{R}^7$, we compute the Serret-Frenet apparatus for OC in $\mathbb{R}^8$.

3. Serret-Frenet Formulas for Octonionic Curves

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^7$ be a space curve and $\{\mathbf{V}_j\}$, $0 \leq j \leq 6$ be the Frenet frame of γ in the Euclidean 7-space $\mathbb{R}^7$. Now, we are going to take the spatial octonions instead of all the Frenet elements. Thus, Frenet elements $\mathbf{V}_j$ can be written as a spatial octonion which is defined by

$$\mathbf{V}_j = \sum_{i=0}^{6} v_j e_j.$$

Definition 3.1. *Let $\mathbb{R}^7$ characterize the Euclidean 7-space with octonionic metric g and $\mathbb{O}_S = \left\{\gamma_{\mathbb{O}} \in \mathbb{O} \;\middle|\; \gamma_{\mathbb{O}} + \bar{\gamma_{\mathbb{O}}} = 0\right\}$ show the spatial octonion set. $\mathbb{R}^7$ is identified with the set of the spatial octonion. The curve $\gamma_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}_S$, $\gamma_{\mathbb{O}}(s) = \sum_{i=1}^{7} \gamma_i(s) e_i$ is called the spatial octonionic curve (SOC).*

Definition 3.2. *If the norm of the first derivative of the SOC is equal to 1, then SOC is called unit speed spatial octonionic curves (USSOC).*

Theorem 3.1. *Let $\gamma_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}_S$ be an USSOC and $\mathbf{V}_0(s) = \gamma_{\mathbb{O}}'(s) = \sum_{i=1}^{7} \gamma_i'(s) e_i$ be unit tangent vector of γ. Then, the following equations are provided*

i) $g\left(\mathbf{V}_0, \mathbf{V}_0'\right) = 0$,

ii) $\mathbf{V}_0' \times \overline{\mathbf{V}_0}$ *is a spatial octonion.*

Proof: Let $\gamma_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}_S$, $\gamma_{\mathbb{O}}(s) = \sum_{i=1}^{7} \gamma_i(s) e_i$ be an USSROC. Since

$\mathbf{V}_0 = \gamma_{\mathbb{O}}'(s) = \sum_{i=1}^{7} \gamma_i'(s) e_i$ has unit length (in other words, $\|\mathbf{V}_0(s)\| = 1$ for all s), we get $\|\mathbf{V}_0\|^2 = g(\mathbf{V}_0, \mathbf{V}_0) = \mathbf{V}_0 \times \overline{\mathbf{V}_0} = 1$. Thus, differentiating with respect to s gives $\mathbf{V}_0' \times \overline{\mathbf{V}_0} + \mathbf{V}_0 \times \left(\overline{\mathbf{V}_0}\right)' = 0$. Since $\mathbf{V}_0 = \gamma'(s) = \sum_{i=1}^{7} \gamma_i'(s) e_i$, we may write $\overline{\mathbf{V}_0} = -\sum_{i=1}^{7} \gamma_i' e_i$ and $\left(\overline{\mathbf{V}_0}\right)' = -\sum_{i=1}^{7} \gamma_i'' e_i$. So, we have $\left(\overline{\mathbf{V}_0}\right)' = \overline{\mathbf{V}_0'}$. Substituting the statement $\left(\overline{\mathbf{V}_0}\right)' = \overline{\mathbf{V}_0'}$ into $\mathbf{V}_0 \times \overline{\mathbf{V}_0} + \mathbf{V}_0 \times \left(\overline{\mathbf{V}_0}\right)'$, we obtain

$$\mathbf{V}_0' \times \overline{\mathbf{V}_0} + \mathbf{V}_0 \times \overline{\mathbf{V}_0'} = 0. \tag{3.1}$$

i) If we multiply on both sides of (3.1) by $\frac{1}{2}$ and put in order it, then we get $g\left(\mathbf{V}_0, \mathbf{V}_0'\right) = \frac{1}{2}\left[\mathbf{V}_0' \times \overline{\mathbf{V}_0} + \mathbf{V}_0 \times \overline{\mathbf{V}_0'}\right] = 0$. Thus, $\mathbf{V}_0'$ is orthogonal to $\mathbf{V}_0$.

ii) By using the definition of the conjugate of the octonion into (3.1), we get $\mathbf{V}_0' \times \overline{\mathbf{V}_0} + \mathbf{V}_0 \times \overline{\mathbf{V}_0'} = \mathbf{V}_0' \times \overline{\mathbf{V}_0} + \overline{\overline{\mathbf{V}_0}} \times \overline{\mathbf{V}_0'} = \mathbf{V}_0' \times \overline{\mathbf{V}_0} + \overline{\left(\mathbf{V}_0' \times \overline{\mathbf{V}_0}\right)} = 0$. Finally, $\mathbf{V}_0' \times \overline{\mathbf{V}_0}$ is a spatial octonion. □

Note 3.1 Let $\gamma_{\mathbb{O}}$ be an USSOC and $\{\mathbf{V}_j\}$, $0 \leq j \leq 6$ be the Frenet frame of SOC in $\mathbb{R}^7$. The Frenet elements of USSOC are given by

$$\mathbf{V}_i \times \mathbf{V}_j = \langle \mathbf{V}_i, \mathbf{V}_j \rangle + \mathbf{V_i} \wedge \mathbf{V_j} = \mathbf{V_k},$$

where $ijk = 012, 034, 065, 135, 146, 236, 254$. On the other hand, we get following table.

Table 1.The Multiplication Table of Unit Octonion Basis Elements

$\times$	$\mathbf{V}_0$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$
$\mathbf{V}_0$	-1	$\mathbf{V_2}$	$-\mathbf{V_1}$	$\mathbf{V_4}$	$-\mathbf{V_3}$	$-\mathbf{V_6}$	$\mathbf{V_5}$
$\mathbf{V_1}$	$-\mathbf{V_2}$	-1	$\mathbf{V}_0$	$\mathbf{V_5}$	$\mathbf{V_6}$	$-\mathbf{V_3}$	$-\mathbf{V_4}$
$\mathbf{V_2}$	$\mathbf{V_1}$	$-\mathbf{V}_0$	-1	$\mathbf{V_6}$	$-\mathbf{V_5}$	$\mathbf{V_4}$	$-\mathbf{V_3}$
$\mathbf{V_3}$	$-\mathbf{V_4}$	$-\mathbf{V_5}$	$-\mathbf{V_6}$	-1	$\mathbf{V}_0$	$\mathbf{V_1}$	$\mathbf{V_2}$
$\mathbf{V_4}$	$\mathbf{V_3}$	$-\mathbf{V_6}$	$\mathbf{V_5}$	$-\mathbf{V}_0$	-1	$-\mathbf{V_2}$	$\mathbf{V_1}$
$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_3}$	$-\mathbf{V_4}$	$-\mathbf{V_1}$	$\mathbf{V_2}$	-1	$-\mathbf{V}_0$
$\mathbf{V_6}$	$-\mathbf{V_5}$	$\mathbf{V_4}$	$\mathbf{V_3}$	$-\mathbf{V_2}$	$-\mathbf{V_1}$	$\mathbf{V}_0$	-1

Theorem 3.2. *Let $\gamma_{\mathbb{O}}$ be an USSOC and $\{\mathbf{V}_j\}$, $0 \leq j \leq 6$ be the Frenet frame of USSOC in $\mathbb{R}^7$. Then the Frenet equations are obtained by*

$$\begin{array}{l} \mathbf{V}_0^{'}(s) = k_1(s)\,\mathbf{V_1}(s) \\ \mathbf{V_m}^{'}(s) = -k_m(s)\,\mathbf{V}_{m-1}(s) + k_{m+1}(s)\,\mathbf{V}_{m+1}(s) \\ \mathbf{V_6}^{'}(s) = -k_6(s)\,\mathbf{V_5}(s), \end{array} \tag{3.2}$$

where $k_i, 1 \leq i \leq 6$, $1 \leq m \leq 5$ curvature functions. The Eq. (3.2) is called Serret-Frenet formulae for the USSOC.

Proof: Since $\mathbf{V}_0^{'} = \sum\limits_{i=1}^{7} \gamma_i^{''} e_i$ is a spatial octonion, we describe the spatial octonion $\mathbf{V_1}$ and the nonnegative scalar function k_1 as follows:

$$\mathbf{V}_0^{'} = k_1 \mathbf{V_1}. \tag{3.3}$$

I. $\mathbf{V_1}^{'}(s) + k_1 \mathbf{V}_0(s)$ is orthogonal to $\mathbf{V_0}$ and $\mathbf{V_1}$. From the definition of Frenet frame of SOC, we get

$$g(\mathbf{V_0}, \mathbf{V_1}) = \frac{1}{2}\left[\mathbf{V_0} \times \overline{\mathbf{V_1}} + \mathbf{V_1} \times \overline{\mathbf{V_0}}\right] = 0. \tag{3.4}$$

Hence, the following equation is obtained from (3.4) by differentiating with respect to s

$$\frac{1}{2}\left[\mathbf{V_0}^{'} \times \overline{\mathbf{V_1}} + \mathbf{V_0} \times \overline{\mathbf{V_1}^{'}} + \mathbf{V_1}^{'} \times \overline{\mathbf{V_0}} + \mathbf{V_1} \times \overline{\mathbf{V_0}^{'}}\right] = 0.$$

Thus, we have

$$\frac{1}{2}\left[\mathbf{V_0}^{'} \times \overline{\mathbf{V_1}} + \mathbf{V_1} \times \overline{\mathbf{V_0}^{'}}\right] + \frac{1}{2}\left[\mathbf{V_0}^{'} \times \overline{\mathbf{V_1}^{'}} + \mathbf{V_1}^{'} \times \overline{\mathbf{V_0}^{'}}\right] = 0.$$

By using the octonionic inner product, then we may write

$$g\left(\mathbf{V_0'},\mathbf{V_1}\right)+g\left(\mathbf{V_1'},\mathbf{V_0}\right)=0\Longrightarrow g\left(\mathbf{V_1'},\mathbf{V_0}\right)=-k_1.$$

Differentiating on both sides of the expression $g(\mathbf{V_1},\mathbf{V_1})=\mathbf{V_1}\times\overline{\mathbf{V_1}}=1$ with respect to s, we get

$$\mathbf{V_1'}\times\overline{\mathbf{V_1}}+\mathbf{V_1}\times\overline{\mathbf{V_1'}}=0\Longrightarrow\frac{1}{2}\left[\mathbf{V_1'}\times\overline{\mathbf{V_1}}+\mathbf{V_1}\times\overline{\mathbf{V_1'}}\right]=\left\langle\mathbf{V_1},\mathbf{V_1'}\right\rangle=0.$$

If we use the last two equations, then we obtain

$$g\left(\mathbf{V_1'}+k_1\mathbf{V_0},\mathbf{V_0}\right)=g\left(\mathbf{V_1'},\mathbf{V_0}\right)+k_1g\left(\mathbf{V_0},\mathbf{V_0}\right)=-k_1+k_1=0,$$

and thus

$$g\left(\mathbf{V_1'}+k_1\mathbf{V_0},\mathbf{V_1}\right)=g\left(\mathbf{V_1'},\mathbf{V_1}\right)+k_1g\left(\mathbf{V_0},\mathbf{V_1}\right)=0.$$

Hence, $\mathbf{V_1'}(s)+k_1\mathbf{V_0}(s)$ is orthogonal to $\mathbf{V_0}$ and $\mathbf{V_1}$, and $(\mathbf{V_0}\times\mathbf{V_1})$ is parallel to $\mathbf{V_1'}(s)+k_1\mathbf{V_0}(s)$. Thus, we have

$$\mathbf{V_1'}(s)+k_1\mathbf{V_0}(s)=\lambda\left(\mathbf{V_0}\times\mathbf{V_1}\right).$$

Since $\mathbf{V_0}\times\mathbf{V_1}=\mathbf{V_2}$, we get

$$g\left(\mathbf{V_1'}+k_1\mathbf{V_0},\mathbf{V_2}\right)=g\left(\lambda\mathbf{V_2},\mathbf{V_2}\right),$$

$$g\left(\mathbf{V_1'},\mathbf{V_2}\right)+k_1g\left(\mathbf{V_0},\mathbf{V_2}\right)=\lambda g\left(\mathbf{V_2},\mathbf{V_2}\right),$$

$$\lambda=g\left(\mathbf{V_1'},\mathbf{V_2}\right)=k_2.$$

Finally, the following equation is obtained by

$$\mathbf{V_1'}(s)=-k_1\mathbf{V_0}(s)+k_2\mathbf{V_2}(s). \tag{3.5}$$

The following cases **II, III, IV, V** can likewise be proved using the techniques of the proof of **I.**

II. Since $\mathbf{V_2'}(s)+k_2\mathbf{V_1}(s)$ is orthogonal to $\mathbf{V_1}$ and $-\mathbf{V_5}$, then ($\mathbf{V_1}\times(-\mathbf{V_5})$) is parallel to $\mathbf{V_2'}(s)+k_2\mathbf{V_1}(s)$. Thus, we have

$$\mathbf{V_2'}(s)=-k_2\mathbf{V_1}(s)+k_3\mathbf{V_3}(s). \tag{3.6}$$

III. Since $\mathbf{V_3'}(s)+k_3\mathbf{V_2}(s)$ is orthogonal to $\mathbf{V_2}$ and $\mathbf{V_5}$, then $(\mathbf{V_2}\times\mathbf{V_5})$ is parallel to $\mathbf{V_3'}(s)+k_3\mathbf{V_2}(s)$. Thus, we get

$$\mathbf{V_3'}(s)=-k_3\mathbf{V_2}(s)+k_4\mathbf{V_4}(s). \tag{3.7}$$

IV. Since $\mathbf{V}_{\mathbf{4}}^{'}(s)+k_4\mathbf{V}_{\mathbf{3}}(s)$ is orthogonal to $\mathbf{V}_{\mathbf{3}}$ and $-\mathbf{V}_{\mathbf{1}}$,then $(\mathbf{V}_{\mathbf{3}}\times(-\mathbf{V}_{\mathbf{1}}))$ is parallel to $\mathbf{V}_{\mathbf{4}}^{'}(s)+k_4\mathbf{V}_{\mathbf{3}}(s)$. Thus, we obtain

$$\mathbf{V}_{\mathbf{4}}^{'}(s)=-k_4\mathbf{V}_{\mathbf{3}}(s)+k_5\mathbf{V}_{\mathbf{5}}(s) \tag{3.8}$$

V. Since $\mathbf{V}_{\mathbf{5}}^{'}(s)+k_5\mathbf{V}_{\mathbf{4}}(s)$ is orthogonal to $\mathbf{V}_{\mathbf{4}}$ and $-\mathbf{V}_{\mathbf{1}}$,then $(\mathbf{V}_{\mathbf{4}}\times(-\mathbf{V}_{\mathbf{1}}))$ is parallel to $\mathbf{V}_{\mathbf{5}}^{'}(s)+k_5\mathbf{V}_{\mathbf{4}}(s)$. Thus, we have

$$\mathbf{V}_{\mathbf{5}}^{'}(s)=-k_5\mathbf{V}_{\mathbf{3}}(s)+k_6\mathbf{V}_{\mathbf{5}}(s) \tag{3.9}$$

VI. Differentiating on both sides of the expression $\mathbf{V}_{\mathbf{5}}=\mathbf{V}_{\mathbf{1}}\wedge\mathbf{V}_3$ with respect to s, we get

$$\mathbf{V}_{\mathbf{5}}^{'}=\mathbf{V}_{\mathbf{1}}^{'}\wedge\mathbf{V}_3+\mathbf{V}_{\mathbf{1}}\wedge\mathbf{V}_{\mathbf{3}}^{'}.$$

Hence, from the last equations we obtain

$$\begin{aligned} g\left(\mathbf{V}_{\mathbf{5}}^{'},\mathbf{V}_{\mathbf{6}}\right) &= g\left(\mathbf{V}_{\mathbf{1}}^{'}\wedge\mathbf{V}_3+\mathbf{V}_{\mathbf{1}}\wedge\mathbf{V}_{\mathbf{3}}^{'},\mathbf{V}_{\mathbf{2}}\wedge\mathbf{V}_3\right) \\ &= g\left(\mathbf{V}_{\mathbf{1}}^{'}\wedge\mathbf{V}_3,\mathbf{V}_{\mathbf{2}}\wedge\mathbf{V}_3\right)+g\left(\mathbf{V}_{\mathbf{1}}\wedge\mathbf{V}_{\mathbf{3}}^{'},\mathbf{V}_{\mathbf{2}}\wedge\mathbf{V}_3\right). \end{aligned}$$

If we use properties vi in the section 2, we may write

$$g\left(\mathbf{V}_{\mathbf{5}}^{'},\mathbf{V}_{\mathbf{6}}\right)=g\left(\mathbf{V}_3\wedge(\mathbf{V}_{\mathbf{2}}\wedge\mathbf{V}_3),\mathbf{V}_{\mathbf{1}}^{'}\right)+g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge(\mathbf{V}_{\mathbf{2}}\wedge\mathbf{V}_3),\mathbf{V}_1\right).$$

Thus, from the properties iii and vii in the section 2, we have

$$\begin{aligned} g\left(\mathbf{V}_{\mathbf{5}}^{'},\mathbf{V}_{\mathbf{6}}\right) &= g\left(\mathbf{V}_3\wedge(-\mathbf{V}_{\mathbf{3}}\wedge\mathbf{V}_2),\mathbf{V}_{\mathbf{1}}^{'}\right)+g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) \\ &= -g\left(\mathbf{V}_3\wedge(\mathbf{V}_{\mathbf{3}}\wedge\mathbf{V}_2),\mathbf{V}_{\mathbf{1}}^{'}\right)+g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) \\ &= -g\left(\langle\mathbf{V}_3,\mathbf{V}_2\rangle\mathbf{V}_{\mathbf{3}}-\langle\mathbf{V}_3,\mathbf{V}_3\rangle\mathbf{V}_2,\mathbf{V}_{\mathbf{1}}^{'}\right)+g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) \\ &= g\left(\mathbf{V}_2,\mathbf{V}_{\mathbf{1}}^{'}\right)+g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) \\ &= k_2+g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right). \end{aligned}$$

Let us calculate

$$\begin{aligned} g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) &= g\left((-k_3\mathbf{V}_{\mathbf{2}}+k_4\mathbf{V}_{\mathbf{4}})\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) \\ &= -k_3 g\left(\mathbf{V}_{\mathbf{2}}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right)+k_4 g\left(\mathbf{V}_{\mathbf{4}}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right) \\ &= -k_3 g\left((-\mathbf{V}_{\mathbf{3}}),\mathbf{V}_1\right)+k_4 g\left(\mathbf{V}_1,\mathbf{V}_1\right) \end{aligned}$$

$$g\left(\mathbf{V}_{\mathbf{3}}^{'}\wedge\mathbf{V}_{\mathbf{6}},\mathbf{V}_1\right)=k_4, \tag{3.10}$$

and thus

$$k_6=g\left(\mathbf{V}_{\mathbf{5}}^{'},\mathbf{V}_{\mathbf{6}}\right)=k_2+k_4. \tag{3.11}$$

Differentiating on both sides of the expression $\mathbf{V_6} = \mathbf{V_2} \times \mathbf{V}_3$ with respect to s, we get

$$\mathbf{V_6}' = \mathbf{V_2}' \times \mathbf{V}_3 + \mathbf{V}_2 \times \mathbf{V_3}'$$

Finally, from the Eqs. (3.6), (3.7), (3.10) and (3.11) we get

$$\begin{aligned}
\mathbf{V_6}' &= \mathbf{V_2}' \times \mathbf{V}_3 + \mathbf{V}_2 \times \mathbf{V_3}' \\
&= \left(-k_2\mathbf{V_1} + k_3\mathbf{V_3}\right) \times \mathbf{V}_3 + \mathbf{V}_2 \times \mathbf{V_3}' \\
&= -k_2\left(\mathbf{V_1} \times \mathbf{V}_3\right) + k_3\left(\mathbf{V}_3 \times \mathbf{V}_3\right) + \mathbf{V}_2 \times \mathbf{V_3}' \\
&= -k_2\mathbf{V_5} - k_3 + \mathbf{V}_2 \times \left(-k_3\mathbf{V}_2 + k_4\mathbf{V_4}\right) \\
&= -k_2\mathbf{V_5} - k_3 - k_3\left(\mathbf{V}_2 \times \mathbf{V}_2\right) + k_4\left(\mathbf{V}_2 \times \mathbf{V_4}\right) \\
&= -k_2\mathbf{V_5} - k_3 + k_3 - k_4\mathbf{V}_5 \\
&= -\left(k_2 + k_4\right)\mathbf{V}_5 \\
&= -k_6\mathbf{V}_5
\end{aligned}$$

□

Corollary 3.3. *Serret-Frenet formulae for the USSROC can be written in matrix notation as follows:*

$$\begin{bmatrix} \mathbf{V_0}' \\ \mathbf{V_1}' \\ \mathbf{V_2}' \\ \mathbf{V_3}' \\ \mathbf{V_4}' \\ \mathbf{V_5}' \\ \mathbf{V_6}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & -k_4 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & -k_5 & 0 & k_6 \\ 0 & 0 & 0 & 0 & 0 & -k_6 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V_1} \\ \mathbf{V_2} \\ \mathbf{V_3} \\ \mathbf{V_4} \\ \mathbf{V_5} \\ \mathbf{V_6} \end{bmatrix}. \tag{3.12}$$

Let us now by using the Serret-Frenet apparatus for SOC, we compute the Serret-Frenet apparatus for OC.

Let $\beta : I \subset \mathbb{R} \to \mathbb{R}^8$ be a space curve and $\{\mathbf{W}_j\}$, $0 \leq j \leq 7$ be the Frenet frame of β in the the Euclidean 8-space, $\mathbb{R}^8$. Now, we are going to take the octonions instead of all the Frenet elements. Thus, Frenet elements $\mathbf{W}_j$ can be written as an octonion which is defined by

$$\mathbf{W}_j = \sum_{i=0}^{7} w_j e_j.$$

Definition 3.3. *Let $\mathbb{R}^8$ characterize the Euclidean 8-space with octonionic metric g and $\mathbb{R}^8$ is identified with the set of the octonion. The curve $\beta_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}$, $\beta_{\mathbb{O}}(s) = \sum\limits_{i=0}^{7} \gamma_i(s)\, e_i$ is called octonionic curve (OC). Note that the vector part of $\beta_{\mathbb{O}}$ is same to SOC $\gamma_{\mathbb{O}}$ in $\mathbb{O}_S$.*

Definition 3.4. *If the norm of the first derivative of the OC is equal to 1, then OC is called the unit speed octonionic curves (USOC).*

Theorem 3.4. *Let $\beta_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}$ be an USOC and* $\mathbf{W}_0(s) = \beta'(s) = \sum_{i=0}^{7} \gamma_i'(s) e_i$ *be unit tangent vector of β. Then, $\mathbf{W}_0'$ is orthogonal to $\mathbf{W}_0$.*

Proof: Let $\beta_{\mathbb{O}} : I \subset \mathbb{R} \to \mathbb{O}$,

$$\beta_{\mathbb{O}}(s) = \sum_{i=1}^{7} \gamma_i(s) e_i, \tag{3.13}$$

be an USOC. Since the tangent $\mathbf{W}_0 = \beta'(s) = \sum_{i=0}^{7} \gamma_i'(s) e_i$ has unit length (in other words, $\|\mathbf{W}_0\| = 1$ for all s), we get

$$\|\mathbf{W}_0\|^2 = g(\mathbf{W}_0, \mathbf{W}_0) = \mathbf{W}_0 \times \overline{\mathbf{W}_0} = 1.$$

Thus, differentiating with respect to s gives

$$\mathbf{W}_0' \times \overline{\mathbf{W}_0} + \mathbf{W}_0 \times \left(\overline{\mathbf{W}_0}\right)' = 0.$$

Since $\mathbf{W}_0 = \sum_{i=0}^{7} \gamma_i'(s) e_i$, we may write $\overline{\mathbf{W}_0} = \gamma_0' - \sum_{i=1}^{7} \gamma_i' e_i$ and so $\left(\overline{\mathbf{W}_0}\right)' = \gamma_0'' - \sum_{i=1}^{7} \gamma_i'' e_i$. So, we have $\left(\overline{\mathbf{W}_0}\right)' = \overline{\mathbf{W}_0'}$. Substituting the statement $\left(\overline{\mathbf{W}_0}\right)' = \overline{\mathbf{W}_0'}$ into the Eq. (3.13), we obtain

$$\mathbf{W}_0' \times \overline{\mathbf{W}_0} + \mathbf{W}_0 \times \overline{\mathbf{W}_0'} = 0.$$

In this case, $\mathbf{W}_0'$ is orthogonal to $\mathbf{W}_0$. □

Theorem 3.5. *Let $\beta_{\mathbb{O}}$ be an USOC and $\{\mathbf{W}_j\}$, $0 \le j \le 7$ be the Frenet frame of USOC in $\mathbb{R}^8$. Then, Frenet equations are obtained by*

$$\begin{aligned}
\mathbf{W}_0'(s) &= K(s)\,\mathbf{W_1}(s) \\
\mathbf{W_1}'(s) &= -K(s)\,\mathbf{W}_0(s) + k_1(s)\,\mathbf{W_2}(s) \\
\mathbf{W_2}'(s) &= -k_1(s)\,\mathbf{W_1}(s) + (k_2 - K)(s)\,\mathbf{W_3}(s) \\
\mathbf{W_3}'(s) &= -(k_2 - K)(s)\,\mathbf{W_2}(s) + k_3(s)\,\mathbf{W_4}(s) \\
\mathbf{W_4}'(s) &= -k_3(s)\,\mathbf{W_3}(s) + (k_4 - K)(s)\,\mathbf{W_5}(s) \\
\mathbf{W_5}'(s) &= -(k_4 - K)(s)\,\mathbf{W_4}(s) + k_5(s)\,\mathbf{W_6}(s) \\
\mathbf{W_6}'(s) &= -k_5(s)\,\mathbf{W_5}(s) + (k_6 + K)(s)\,\mathbf{W_7}(s) \\
\mathbf{W_7}'(s) &= -(k_6 + K)(s)\,\mathbf{W_6}(s),
\end{aligned} \tag{3.14}$$

where $\mathbf{W_1} = \mathbf{V}_0 \times \mathbf{W}_0$, $\mathbf{W_2} = \mathbf{V_1} \times \mathbf{W}_0$, $\mathbf{W_3} = \mathbf{V_2} \times \mathbf{W}_0$, $\mathbf{W_4} = \mathbf{V_3} \times \mathbf{W}_0$, $\mathbf{W_5} = \mathbf{V_4} \times \mathbf{W}_0$, $\mathbf{W_6} = \mathbf{V_5} \times \mathbf{W}_0$, $\mathbf{W_7} = \mathbf{V_6} \times \mathbf{W}_0$, $K = \left\|\mathbf{W}_0'(s)\right\|$.

Proof: Let us assume that

$$\mathbf{W}_0' = K\mathbf{W_1},\ K = \left\|\mathbf{W}_0'(s)\right\|,\ \|\mathbf{W_1}\| = 1. \tag{3.15}$$

Hence, substituting Eq. (3.15) into the $\mathbf{W}_0' \times \overline{\mathbf{W}_0} + \mathbf{W}_0 \times \overline{\mathbf{W}_0'} = 0$**,** we get

$$\begin{aligned} \left(K\mathbf{W_1} \times \overline{\mathbf{W}_0}\right) + \left(\mathbf{W}_0 \times \overline{K\mathbf{W_1}}\right) &= 0 \\ K\left(\mathbf{W_1} \times \overline{\mathbf{W}_0} + \mathbf{W}_0 \times \overline{\mathbf{W_1}}\right) &= 0. \end{aligned}$$

On the other hand, $\mathbf{W_1}$ is orthogonal to $\mathbf{W}_0$. So, we have $g(\mathbf{W}_0, \mathbf{W_1}) = 0$. From the rules of conjugate of the octonion, we obtain

$$\mathbf{W_1} \times \overline{\mathbf{W}_0} + \mathbf{W}_0 \times \overline{\mathbf{W_1}} = \mathbf{W_1} \times \overline{\mathbf{W}_0} + \overline{\mathbf{W_1} \times \overline{\mathbf{W}_0}} = 0.$$

Thus, $\mathbf{W_1} \times \overline{\mathbf{W}_0}$ is a spatial octonion.

Since $\mathbf{V}_0$ and $\mathbf{W_1} \times \overline{\mathbf{W}_0}$ are the spatial octonions and they have unit magnitude, we may choose $\mathbf{V}_0$ as follows:

$$\mathbf{V}_0 = \mathbf{W_1} \times \overline{\mathbf{W}_0}$$

On the other hand, we get

$$\begin{aligned} \mathbf{V}_0 \times \mathbf{W}_0 &= \left(\mathbf{W_1} \times \overline{\mathbf{W}_0}\right) \times \mathbf{W}_0 \\ &= \left(\mathbf{W_1} \times \mathbf{W}_0^{-1}\right) \times \mathbf{W}_0 \\ &= \mathbf{W_1}, \end{aligned}$$

and thus

$$\mathbf{W_1} = \mathbf{V}_0 \times \mathbf{W}_0. \tag{3.16}$$

By differentiating the last equation with respect to s and using the Eqs. (3.3) and (3.15) in the last equation, then we have

$$\begin{aligned} \mathbf{W_1}' &= \mathbf{V}_0' \times \mathbf{W}_0 + \mathbf{V}_0 \times \mathbf{W}_0' \\ &= k_1\mathbf{V_1} \times \mathbf{W}_0 + \mathbf{V}_0 \times K\mathbf{W_1} \\ &= k_1\mathbf{W_2} + K\left(\mathbf{V}_0 \times \mathbf{W_1}\right),\ \mathbf{W_2} = \mathbf{V_1} \times \mathbf{W}_0 \\ &= k_1\mathbf{W_2} + K\left(\mathbf{V}_0 \times \left(\mathbf{V}_0 \times \mathbf{W}_0\right)\right) \\ &= -K\mathbf{W}_0 + k_1\mathbf{W_2}, \end{aligned}$$

and thus

$$\mathbf{W_1}' = -K\mathbf{W}_0 + k_1\mathbf{W_2},\ \ \mathbf{W_2} = \mathbf{V_1} \times \mathbf{W}_0. \tag{3.17}$$

Let us give some informations about $\mathbf{W_2}$ as follows:

i) $\|\mathbf{W_2}\|^2 = 1.$

ii) $\mathbf{W_2}(s)$ is a smooth octonion function of s and $\mathbf{W}_0$,$\mathbf{W_1}$ and $\mathbf{W_2}$ are mutually g orthogonal since $\mathbf{V_0}$ and $\mathbf{V_1}$ are so.

By differentiating $\mathbf{W_2}$ given by Eq. (3.17) and substituting the Eqs. (3.5), (3.15) and (3.16) into this equation, we get

$$\mathbf{W_2}' = -k_1\mathbf{W_1}(s) + (k_2 - K)\mathbf{W_3},\ \mathbf{W_3} = \mathbf{V_2} \times \mathbf{W}_0. \tag{3.18}$$

Let us give some informations about $\mathbf{W_3}$ as follows:

i) $\|\mathbf{W_3}\|^2 = 1$.

ii) $\mathbf{W_3}(s)$ is a smooth octonion function of s and $\mathbf{W}_0$,$\mathbf{W_1}$,$\mathbf{W_2}$,and $\mathbf{W_3}$ are mutually g orthogonal since $\mathbf{V}_0$,$\mathbf{V_1}$ and $\mathbf{V_2}$ are so.

By differentiating $\mathbf{W_3}$ given by Eq. (3.18) and substituting the Eqs. (3.6), (3.15), (3.16) and (3.17) into this equation, we get

$$\mathbf{W_3}' = -(k_2 - K)\mathbf{W_2} + k_3\mathbf{W_4},\ \mathbf{W_4} = \mathbf{V_3} \times \mathbf{W}_0. \tag{3.19}$$

Let us give some informations about $\mathbf{W_4}$ as follows:

i) $\|\mathbf{W_4}\|^2 = 1$.

ii) $\mathbf{W_4}(s)$ is a smooth octonion function of s and $\mathbf{W}_0$,$\mathbf{W_1}$,$\mathbf{W_2}$,$\mathbf{W_3}$,and $\mathbf{W_4}$ are mutually g orthogonal since $\mathbf{V}_0$,$\mathbf{V_1}$,$\mathbf{V_2}$ and $\mathbf{V_3}$ are so.

By differentiating $\mathbf{W_4}$ given by Eq. (3.19) and substituting the Eqs. (3.7), (3.15), (3.17) and (3.18) into this equation, we get

$$\mathbf{W_4}' = -k_3\mathbf{W_3} + (k_4 - K)\mathbf{W_5},\ \mathbf{W_5} = \mathbf{V_4} \times \mathbf{W}_0. \tag{3.20}$$

Let us give some informations about $\mathbf{W_5}$ as follows:

i) $\|\mathbf{W_5}\|^2 = 1$.

ii) $\mathbf{W_5}(s)$ is a smooth octonion function of s and $\mathbf{W}_0$,$\mathbf{W_1}$,$\mathbf{W_2}$,$\mathbf{W_3}$,$\mathbf{W_4}$,and $\mathbf{W_5}$ are mutually g orthogonal since $\mathbf{V}_0$,$\mathbf{V_1}$,$\mathbf{V_2}$,$\mathbf{V_3}$ and $\mathbf{V_4}$ are so.

By differentiating $\mathbf{W_5}$ given by Eq. (3.20) and substituting the Eqs. (3.8), (3.15), (3.18) and (3.19) into this equation, we get

$$\mathbf{W_5}' = -(k_4 - K)\mathbf{W_4} + k_5\mathbf{W_6},\ \mathbf{W_6} = \mathbf{V_5} \times \mathbf{W}_0. \tag{3.21}$$

Let us give some informations about $\mathbf{W_6}$ as follows:

i) $\|\mathbf{W_6}\|^2 = 1$.

ii) $\mathbf{W_6}(s)$ is a smooth octonion function of s and $\mathbf{W}_0$,$\mathbf{W_1}$,$\mathbf{W_2}$,$\mathbf{W_3}$,$\mathbf{W_4}$,$\mathbf{W_5}$,and $\mathbf{W_6}$ are mutually g orthogonal since $\mathbf{V}_0$,$\mathbf{V_1}$,$\mathbf{V_2}$,$\mathbf{V_3}$,$\mathbf{V_4}$ and $\mathbf{V_5}$ are so.

By differentiating $\mathbf{W_6}$ given by Eq. (3.21) and substituting the Eqs. (3.9), (3.15), (3.19) and (3.20) into this equation, we get

$$\mathbf{W_6}' = -k_5\mathbf{W_5} + (k_6 + K)\mathbf{W_7},\ \mathbf{W_7} = \mathbf{V_6} \times \mathbf{W}_0. \tag{3.22}$$

Let us give some informations about $\mathbf{W_7}$ as follows:

i) $\|\mathbf{W_7}\|^2 = 1$.

ii) $\mathbf{W_7}(s)$ is a smooth octonion function of s and $\mathbf{W}_0$,$\mathbf{W_1}$,$\mathbf{W_2}$,$\mathbf{W_3}$,$\mathbf{W_4}$,$\mathbf{W_5}$, $\mathbf{W_6}$, and $\mathbf{W_7}$ are mutually g orthogonal since $\mathbf{V}_0$,$\mathbf{V_1}$,$\mathbf{V_2}$,$\mathbf{V_3}$,$\mathbf{V_4}$,$\mathbf{V_5}$ and $\mathbf{V_6}$ are so.

Finally, by differentiating $\mathbf{W_7}$ given by Eq. (3.22) and substituting the Eqs. (3.12) and (3.15) we get

$$\mathbf{W_7}' = -\left(k_6 + K\right)\mathbf{W_6}. \tag{3.23}$$

Serret-Frenet formulae for the USOC can be written in matrix notation as follows:

$$\frac{d}{ds}\begin{bmatrix}\mathbf{W}_0\\ \mathbf{W_1}\\ \mathbf{W_2}\\ \mathbf{W_3}\\ \mathbf{W_4}\\ \mathbf{W_5}\\ \mathbf{W_6}\\ \mathbf{W_7}\end{bmatrix} = \mathbf{W}\cdot\begin{bmatrix}\mathbf{W}_0\\ \mathbf{W_1}\\ \mathbf{W_2}\\ \mathbf{W_3}\\ \mathbf{W_4}\\ \mathbf{W_5}\\ \mathbf{W_6}\\ \mathbf{W_7}\end{bmatrix},$$

where

$$\mathbf{W} = \begin{bmatrix} 0 & K & 0 & 0 & 0 & 0 & 0 & 0\\ -K & 0 & k_1 & 0 & 0 & 0 & 0 & 0\\ 0 & -k_1 & 0 & (k_2 - K) & 0 & 0 & 0 & 0\\ 0 & 0 & -(k_2 - K) & 0 & k_3 & 0 & 0 & 0\\ 0 & 0 & 0 & -k_3 & 0 & (k_4 - K) & 0 & 0\\ 0 & 0 & 0 & 0 & -(k_4 - K) & 0 & k_5 & 0\\ 0 & 0 & 0 & 0 & 0 & -k_5 & 0 & (k_6 + K)\\ 0 & 0 & 0 & 0 & 0 & 0 & -(k_6 + K) & 0 \end{bmatrix}.$$

This is the Serret Frenet formulae for USOC $\beta_{\mathbb{O}}$ in $\mathbb{R}^8$.

$$\left\{\mathbf{W}_0,\mathbf{W_1},\mathbf{W_2},\mathbf{W_3},\mathbf{W_4},\mathbf{W_5},\mathbf{W_6},\mathbf{W_7},K,k_1,\left(k_2 - K\right),k_3,\left(k_4 - K\right),k_5,\left(k_6 + K\right)\right\}$$

Serret-Frenet apparatus for the USOC $\beta_{\mathbb{O}}$ in $\mathbb{R}^8$. □

4. Conclusion

We obtain the Serret-Frenet formulae and the Serret-Frenet apparatus for the ROC $\beta_{\mathbb{O}}$ by making use of the Serret-Frenet formulae for a SROC $\gamma_{\mathbb{O}}$ in $\mathbb{R}^7$. This curve is so chosen that the unit tangent to it is $\mathbf{V}_0\left(s\right)$ is given by (3.16). It should be noted here that the second curvature of $\beta_{\mathbb{O}}$ is first curvature of $\gamma_{\mathbb{O}}$, fourth curvature of $\beta_{\mathbb{O}}$ is third curvature of γ, sixth curvature of $\beta_{\mathbb{O}}$ is fifth curvature of γ. Note that the third curvature of $\beta_{\mathbb{O}}$ is $(k_2 - K)$, where k_2 is the second curvature of the curve $\gamma_{\mathbb{O}}$ and K is the first curvature of $\beta_{\mathbb{O}}$, fifth curvature of $\beta_{\mathbb{O}}$ is $(k_4 - K)$, where k_4 is the fourth curvature of the curve $\gamma_{\mathbb{O}}$ and K is the first curvature of $\beta_{\mathbb{O}}$, sixth curvature of $\beta_{\mathbb{O}}$ is $(k_6 + K)$, where k_6 is the sixth curvature of the curve $\gamma_{\mathbb{O}}$ and K is the first curvature of $\beta_{\mathbb{O}}$. Also, the sum of second curvature and fourth curvature of $\gamma_{\mathbb{O}}$ is sixth curvature of $\gamma_{\mathbb{O}}$. As can be easily seen, classical methods of elemantary differential geometry do note give us the technique to determine the SROC $\gamma_{\mathbb{O}}$ in $\mathbb{R}^7$ corresponding to $\beta_{\mathbb{O}}$ in $\mathbb{R}^8$. Space curves in 7 and 8 Euclidean space are defined by with this study. This paper is important from this point. In view of the quaternionic curves, there are a lot of papers about quaternionic curves. After these defining octonionic curves, a lot of paper about octonionic curves can be studied.

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Özcan Bektaş
Department of Mathematics,
Recep Tayyip Erdogan University, Rize
Turkey.
E-mail address: ozcan.bektas@erdogan.edu.tr

and

Salim Yüce
Department of Mathematics,
Yildiz Technical University, Instanbul
Turkey.
E-mail address: sayuce@yildiz.edu.tr