Representations by octonary quadratic forms with coefficients 1, 3 or 9

Article · December 2015

DOI: 10.1142/S1793042115501080

2 authors:

Ayşe Alaca
Carleton University

Mücahide Nesibe Kesicioğlu
Recep Tayyip Erdoğan Üniversitesi

REPRESENTATIONS BY OCTONARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 3 OR 6

AYŞE ALACA AND M. NESIBE KESICIOĞLU

Abstract. Using modular forms we determine formulas for the number of representations of a positive integer by diagonal octonary quadratic forms with coefficients 1, 2, 3 or 6.

Key words and phrases: octonary quadratic forms, representations, theta functions, Dedekind eta function, eta quotients, Eisenstein series, Eisenstein forms, modular forms, cusp forms.

2010 Mathematics Subject Classification: 11E25, 11F27

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers and complex numbers respectively. For $a_1, \ldots, a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a_1, \ldots, a_8; n) := \text{card}\{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \mid n = a_1x_1^2 + \cdots + a_8x_8^2\}.$$

Clearly $N(a_1, \ldots, a_8; 0) = 1$. Without loss of generality we may suppose that

$$a_1 \leq \cdots \leq a_8 \text{ and } \gcd(a_1, \ldots, a_8) = 1.$$

Formulae for $N(a_1, \ldots, a_8; n)$ for the octonary quadratic forms

$$\begin{aligned}(1.1) \qquad &(x_1^2 + \cdots + x_i^2) + 2(x_{i+1}^2 + \cdots + x_{i+j}^2) \\ &+3(x_{i+j+1}^2 + \cdots + x_{i+j+k}^2) + 6(x_{i+j+k+1}^2 + \cdots + x_{i+j+k+l}^2)\end{aligned}$$

with $i + j + k + l = 8$ under the conditions

$$i \equiv j \equiv k \equiv l \equiv 0 \pmod 2 \text{ or } i \equiv j \equiv k \equiv l \equiv 1 \pmod 2$$

appeared in literature. See [1], [2], [3], [4], [6], [9] and [11]. For convenience, we write (i, j, k, l) to denote an octonary quadratic form given by (1.1), and we write $N(1^i, 2^j, 3^k, 6^l; n)$ to denote the number of representations of n by the octonary quadratic form (i, j, k, l).

In this paper we determine a formula for $N(1^i, 2^j, 3^k, 6^l; n)$ for each of the octonary quadratic forms (i, j, k, l) when some of the i, j, k or l have different parities from the others. There are one hundred and twelve such cases, and all of them are listed in Table 1. This paper completes the representations of a positive integer by diagonal octonary quadratic forms with coefficients 1, 2, 3 or 6.

TABLE 1. Octonary quadratic forms (i, j, k, l)

(i,j,k,l)	(i,j,k,l)	(i,j,k,l)
$(0,2,1,5)$, $(2,0,5,1)$	$(0,1,2,5)$, $(2,1,4,1)$	$(0,1,1,6)$, $(2,1,1,4)$
$(0,2,3,3)$, $(2,2,1,3)$	$(0,1,4,3)$, $(2,3,0,3)$	$(0,1,3,4)$, $(2,1,3,2)$
$(0,2,5,1)$, $(2,2,3,1)$	$(0,1,6,1)$, $(2,3,2,1)$	$(0,1,5,2)$, $(2,1,5,0)$
$(0,4,1,3)$, $(2,4,1,1)$	$(0,3,2,3)$, $(2,5,0,1)$	$(0,1,7,0)$, $(2,3,1,2)$
$(0,4,3,1)$, $(3,1,0,4)$	$(0,3,4,1)$, $(3,0,1,4)$	$(0,3,1,4)$, $(2,3,3,0)$
$(0,6,1,1)$, $(3,1,2,2)$	$(0,5,2,1)$, $(3,0,3,2)$	$(0,3,3,2)$, $(2,5,1,0)$
$(1,1,0,6)$, $(3,1,4,0)$	$(1,0,1,6)$, $(3,0,5,0)$	$(0,3,5,0)$, $(3,0,0,5)$
$(1,1,2,4)$, $(3,3,0,2)$	$(1,0,3,4)$, $(3,2,1,2)$	$(0,5,1,2)$, $(3,0,2,3)$
$(1,1,4,2)$, $(3,3,2,0)$	$(1,0,5,2)$, $(3,2,3,0)$	$(0,5,3,0)$, $(3,0,4,1)$
$(1,1,6,0)$, $(3,5,0,0)$	$(1,0,7,0)$, $(3,4,1,0)$	$(0,7,1,0)$, $(3,2,0,3)$
$(1,3,0,4)$, $(4,0,1,3)$	$(1,2,1,4)$, $(4,1,0,3)$	$(1,0,0,7)$, $(3,2,2,1)$
$(1,3,4,0)$, $(4,0,3,1)$	$(1,2,3,2)$, $(4,1,2,1)$	$(1,0,2,5)$, $(3,4,0,1)$
$(1,3,2,2)$, $(4,2,1,1)$	$(1,2,5,0)$, $(4,3,0,1)$	$(1,0,4,3)$, $(4,1,1,2)$
$(1,5,0,2)$, $(5,1,0,2)$	$(1,4,1,2)$, $(5,0,1,2)$	$(1,0,6,1)$, $(4,1,3,0)$
$(1,5,2,0)$, $(5,1,2,0)$	$(1,4,3,0)$, $(5,0,3,0)$	$(1,2,0,5)$, $(4,3,1,0)$
$(1,7,0,0)$, $(5,3,0,0)$	$(1,6,1,0)$, $(5,2,1,0)$	$(1,2,2,3)$, $(5,0,0,3)$
$(2,0,1,5)$, $(6,0,1,1)$	$(2,1,0,5)$, $(6,1,0,1)$	$(1,2,4,1)$, $(5,0,2,1)$
$(2,0,3,3)$, $(7,1,0,0)$	$(2,1,2,3)$, $(7,0,1,0)$	$(1,4,0,3)$, $(5,2,0,1)$
		$(1,4,2,1)$, $(6,1,1,0)$
		$(1,6,0,1)$, $(7,0,0,1)$

2. PRELIMINARY RESULTS

For $q \in \mathbb{C}$ with $|q| < 1$ we define

$$F(q) := \prod_{n=1}^{\infty}(1 - q^n). \tag{2.1}$$

Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

We note that

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_8; n)q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_8}). \tag{2.2}$$

The infinite product representation of $\varphi(q)$ is due to Jacobi [6], namely

$$\varphi(q) = \frac{F^5(q^2)}{F^2(q)F^2(q^4)}. \tag{2.3}$$

The Dedekind eta function $\eta(z)$ is defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{2.4}$$

Throughout the remainder of the paper we take $q = q(z) := e^{2\pi i z}$ with $z \in \mathbb{H}$ so that $|q| < 1$. By (2.1) and (2.4) we have

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} F(q). \tag{2.5}$$

An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z), \tag{2.6}$$

where δ runs through a finite set of positive integers and the exponents r_δ are non-zero integers. By taking N to be the least common multiple of the δ's we can write the eta quotient (2.6) as

$$f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z), \tag{2.7}$$

where some of the exponents r_δ may be 0.

Let $N \in \mathbb{N}$ and χ be a Dirichlet character of modulus dividing N and $\Gamma_0(N)$ the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Let $k \in \mathbb{Z}$. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. It is known (see for example [13, p. 83] and [12]) that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{2.8}$$

We use the following theorem to determine if certain eta quotients are modular forms. See [7, Theorem 5.7, p. 99], [8, Corollary 2.3, p. 37], [5, p. 174] and [10].

Theorem 2.1. ***(Ligozat)*** *Let $N \in \mathbb{N}$. Let $f(z) = \prod_{1\le\delta|N} \eta^{r_\delta}(\delta z)$ be an eta quotient and $s = \prod_{1\le\delta|N} \delta^{r_\delta}$. Suppose that $k = \frac{1}{2}\sum_{1\le\delta|N} r_\delta$ is an integer. If $f(z)$ satisfies the conditions*

(L1) $\sum_{1\le\delta|N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(L2) $\sum_{1\le\delta|N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(L3) *for each $d \mid N$,* $\sum_{1\le\delta|N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} \ge 0$,

then $f(z) \in M_k(\Gamma_0(N), \chi)$, where χ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right)$$

In addition to the above conditions if $f(z)$ also satisfies the condition

(L4) *for each $d \mid N$,* $\sum_{1\le\delta|N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} > 0$,

then $f(z) \in S_k(\Gamma_0(N), \chi)$.

Let ψ_1 and ψ_2 be Dirichlet characters. For $n \in \mathbb{N}$ we define $\sigma_{(3,\psi_1,\psi_2)}(n)$ by

$$\sigma_{(3,\psi_1,\psi_2)}(n) := \sum_{1\le m|n} \psi_2(m)\psi_1(n/m)m^3. \tag{2.9}$$

If $n \notin \mathbb{N}$, we set $\sigma_{(3,\psi_1,\psi_2)}(n) = 0$. If ψ_1 and ψ_2 are trivial characters then $\sigma_{(3,\psi_1,\psi_2)}(n)$ coincides with the sum of divisors function

$$\sigma_3(n) = \sum_{1\le m|n} m^3.$$

Let χ_0 denote the trivial character. For $m \in \mathbb{Z}$, we define six characters by

$$\begin{cases} \chi_1(m) = \left(\frac{-8}{m}\right), \ \chi_2(m) = \left(\frac{-4}{m}\right), \ \chi_3(m) = \left(\frac{-3}{m}\right), \\ \chi_4(m) = \left(\frac{8}{m}\right), \ \chi_5(m) = \left(\frac{12}{m}\right), \ \chi_6(m) = \left(\frac{24}{m}\right). \end{cases} \tag{2.10}$$

We define the Eisenstein series $E_{4,\chi_0,\chi_4}(q)$, $E_{4,\chi_0,\chi_5}(q)$, $E_{4,\chi_0,\chi_6}(q)$, $E_{4,\chi_1,\chi_3}(q)$, $E_{4,\chi_2,\chi_3}(q)$, $E_{4,\chi_3,\chi_1}(q)$, $E_{4,\chi_4,\chi_0}(q)$, $E_{4,\chi_5,\chi_0}(q)$ and , $E_{4,\chi_6,\chi_0}(q)$ by

$$E_{4,\chi_0,\chi_4}(q) := \frac{11}{2} + \sum_{n=1}^{\infty} \sigma_{(3,\chi_0,\chi_4)}(n)q^n, \tag{2.11}$$

(2.12) $$E_{4,\chi_0,\chi_5}(q) := 23 + \sum_{n=1}^{\infty} \sigma_{(3,\chi_0,\chi_5)}(n)q^n,$$

(2.13) $$E_{4,\chi_0,\chi_6}(q) := 261 + \sum_{n=1}^{\infty} \sigma_{(3,\chi_0,\chi_6)}(n)q^n,$$

(2.14) $$E_{4,\chi_1,\chi_3}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_1,\chi_3)}(n)q^n,$$

(2.15) $$E_{4,\chi_2,\chi_3}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_2,\chi_3)}(n)q^n,$$

(2.16) $$E_{4,\chi_3,\chi_1}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_3,\chi_1)}(n)q^n,$$

(2.17) $$E_{4,\chi_3,\chi_2}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_3,\chi_2)}(n)q^n,$$

(2.18) $$E_{4,\chi_4,\chi_0}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_4,\chi_0)}(n)q^n,$$

(2.19) $$E_{4,\chi_5,\chi_0}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_5,\chi_0)}(n)q^n,$$

(2.20) $$E_{4,\chi_6,\chi_0}(q) := \sum_{n=1}^{\infty} \sigma_{(3,\chi_6,\chi_0)}(n)q^n.$$

3. Main results

We define the eta quotients $A_k(q)$, $B_k(q)$ and $C_k(q)$ as in Table 2. We define the integers $a_k(n), b_k(n)$ and $c_k(n)$ $(n \in \mathbb{N})$ by

(3.1) $$A_k(q) = \sum_{n=1}^{\infty} a_k(n)q^n, \ (1 \leq k \leq 10),$$

(3.2) $$B_k(q) = \sum_{n=1}^{\infty} b_k(n)q^n, \ 1 \leq k \leq 8),$$

(3.3) $$C_k(q) = \sum_{n=1}^{\infty} c_k(n)q^n, \ (1 \leq k \leq 10).$$

We note that the eta quotients in Table 2 are constructed by using MAPLE in a way that they satisfy the conditions of Theorem 2.1 for $N = 24$ and $k = 4$. We deduce from [13, Sec. 6.3, p. 98] that

(3.4) $$\dim(S_4(\Gamma_0(24), \chi_4) = 10,$$

Table 2. Eta quotients $A_k(q), B_k(q)$ and $C_k(q)$

k	$A_k(q)$	$B_k(q)$	$C_k(q)$
1	$\frac{\eta^2(3z)\eta^2(4z)\eta^5(6z)\eta^2(8z)}{\eta^3(12z)}$	$\frac{\eta^4(3z)\eta^2(6z)\eta^3(8z)}{\eta(24z)}$	$\frac{\eta^7(6z)\eta^3(8z)\eta^3(12z)}{\eta^2(3z)\eta^3(24z)}$
2	$\frac{\eta^2(3z)\eta^5(4z)\eta(6z)\eta(24z)}{\eta(8z)}$	$\frac{\eta^4(6z)\eta^2(8z)\eta^5(12z)}{\eta(4z)\eta^2(24z)}$	$\frac{\eta^2(3z)\eta^7(4z)\eta^4(12z)}{\eta^3(6z)\eta^2(8z)}$
3	$\frac{\eta^5(2z)\eta(3z)\eta(12z)\eta^2(24z)}{\eta(z)}$	$\frac{\eta^4(3z)\eta^2(4z)\eta^2(6z)\eta^3(24z)}{\eta(8z)\eta^2(12z)}$	$\frac{\eta^2(3z)\eta(6z)\eta^6(8z)\eta^2(12z)}{\eta^3(4z)}$
4	$\frac{\eta^2(3z)\eta(6z)\eta^4(8z)\eta(12z)\eta^2(24z)}{\eta^2(4z)}$	$\frac{\eta^4(6z)\eta^3(8z)\eta^3(24z)}{\eta^2(12z)}$	$\frac{\eta^2(3z)\eta^3(8z)\eta^5(12z)\eta(24z)}{\eta^3(6z)}$
5	$\frac{\eta^2(3z)\eta(4z)\eta(8z)\eta^4(12z)\eta^3(24z)}{\eta^3(6z)}$	$\eta^3(4z)\eta(12z)\eta^4(24z)$	$\frac{\eta^2(3z)\eta(6z)\eta^2(8z)\eta^4(24z)}{\eta(4z)}$
6	$\frac{\eta^2(3z)\eta(6z)\eta^6(24z)}{\eta(12z)}$	$\frac{\eta^2(4z)\eta^4(6z)\eta^7(24z)}{\eta(8z)\eta^4(12z)}$	$\frac{\eta^2(3z)\eta^2(4z)\eta^3(12z)\eta^5(24z)}{\eta^3(6z)\eta(8z)}$
7	$\frac{\eta^2(3z)\eta^3(4z)\eta^2(12z)\eta^7(24z)}{\eta^3(6z)\eta^3(8z)}$	$\frac{\eta^5(4z)\eta^8(24z)}{\eta^4(8z)\eta(12z)}$	$\frac{\eta^2(3z)\eta(4z)\eta(6z)\eta^8(24z)}{\eta^2(8z)\eta^2(12z)}$
8	$\frac{\eta^5(2z)\eta(3z)\eta^3(12z)\eta^8(24z)}{\eta(z)\eta^2(4z)\eta^4(6z)\eta^2(8z)}$	$\frac{\eta^4(2z)\eta^7(24z)}{\eta(8z)\eta^2(12z)}$	$\frac{\eta(z)\eta(6z)\eta(12z)\eta^8(24z)}{\eta(3z)\eta^2(8z)}$
9	$\frac{\eta^2(3z)\eta^6(4z)\eta^5(12z)}{\eta^3(6z)\eta^2(24z)}$		$\frac{\eta^2(2z)\eta^6(3z)\eta(4z)\eta^2(8z)}{\eta^3(6z)}$
10	$\frac{\eta^2(4z)\eta^7(6z)\eta^2(8z)\eta^4(24z)}{\eta^2(3z)\eta^5(12z)}$		$\frac{\eta^2(3z)\eta^3(4z)\eta^5(6z)\eta^2(24z)}{\eta^4(12z)}$

$$\dim(S_4(\Gamma_0(24), \chi_5) = 8, \tag{3.5}$$

$$\dim(S_4(\Gamma_0(24), \chi_6) = 10. \tag{3.6}$$

Theorem 3.1. (a) *$\{A_k(q) \mid 1 \le k \le 10\}$ is a basis for $S_4(\Gamma_0(24), \chi_4)$.*

(b) *$\{E_{4,\chi_0,\chi_4}(q^t), E_{4,\chi_4,\chi_0}(q^t) \mid t = 1, 3\}$ is a basis for $E_4(\Gamma_0(24), \chi_4)$.*

(c) *$\{E_{4,\chi_0,\chi_4}(q^t), E_{4,\chi_4,\chi_0}(q^t) \mid t = 1, 3\} \cup \{A_k(q) \mid 1 \le k \le 10\}$ is a basis for $M_4(\Gamma_0(24), \chi_4)$.*

Proof. (a) By Theorem 2.1, $A_k(q)$ $(1 \le k \le 10)$ are in $S_4(\Gamma_0(24), \chi_4)$. Then the assertion follows from (3.4) as there is no linear relationship among the $A_k(q)$ $(1 \le k \le 10)$.

(b) The assertion follows from [13, Theorem 5.9, p. 88] with $\epsilon = \chi_4$ and $\chi, \psi \in \{\chi_0, \chi_4\}$.

(c) The assertion follows from (a), (b) and (2.8). ■

Theorem 3.2. (a) *$\{B_k(q) \mid 1 \le k \le 8\}$ is a basis for $S_4(\Gamma_0(24), \chi_5)$.*

(b) *$\{E_{4,\chi_0,\chi_5}(q^t), E_{4,\chi_2,\chi_3}(q^t), E_{4,\chi_3,\chi_2}(q^t), E_{4,\chi_5,\chi_0}(q^t) \mid t = 1, 2\}$ is a basis for $E_4(\Gamma_0(24), \chi_5)$.*

(c) *$\{E_{4,\chi_0,\chi_5}(q^t), E_{4,\chi_2,\chi_3}(q^t), E_{4,\chi_3,\chi_2}(q^t), E_{4,\chi_5,\chi_0}(q^t) \mid t = 1, 2\} \cup \{B_k(q) \mid 1 \le k \le 8\}$ is a basis for $M_4(\Gamma_0(24), \chi_5)$.*

Proof. (a) By Theorem 2.1, $B_k(q)$ $(1 \le k \le 8)$ are in $S_4(\Gamma_0(24), \chi_5)$. Then the assetion follows from (3.5) as there is no linear relationship among the $B_k(q)$ $(1 \le k \le 8)$.

(b) The assertion follows from [13, Theorem 5.9, p. 88] with $\epsilon = \chi_5$ and $\chi, \psi \in \{\chi_0, \chi_2, \chi_3, \chi_5\}$.

(c) The assertion follows from (a), (b) and (2.8). ■

Theorem 3.3. (a) $\{C_k(q) \mid 1 \leq k \leq 10\}$ *is a basis for* $S_4(\Gamma_0(24), \chi_6)$.

(b) $\{E_{4,\chi_0,\chi_6}(q), E_{4,\chi_1,\chi_3}(q), E_{4,\chi_3,\chi_1}(q), E_{4,\chi_6,\chi_0}(q)\}$ *is a basis for* $E_4(\Gamma_0(24), \chi_6)$.

(c) $\{E_{4,\chi_0,\chi_6}(q), E_{4,\chi_1,\chi_3}(q), E_{4,\chi_3,\chi_1}(q), E_{4,\chi_6,\chi_0}(q)\} \cup \{C_k(q) \mid 1 \leq k \leq 10\}$ *is a basis for* $M_4(\Gamma_0(24), \chi_6)$.

Proof. (a) By Theorem 2.1, $C_k(q)$ $(1 \leq k \leq 8)$ are in $S_4(\Gamma_0(24), \chi_6)$. Then the assetion follows from (3.6) as there is no linear relationship among the $C_k(q)$ $(1 \leq k \leq 10)$.

(b) The assertion follows from [13, Theorem 5.9, p. 88] with $\epsilon = \chi_6$ and $\chi, \psi \in \{\chi_0, \chi_1, \chi_3, \chi_6\}$.

(c) The assertion follows from (a), (b) and (2.8). ■

Theorem 3.4. *Let* (i, j, k, l) *be any of the octonary quadratic forms listed in the first column of* Table 1. *Then*

$$\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6) = x_1 E_{4,\chi_0,\chi_4}(q) + x_2 E_{4,\chi_4,\chi_0}(q) + x_3 E_{4,\chi_0,\chi_4}(q^3) + x_4 E_{4,\chi_4,\chi_0}(q^3) + \sum_{i=1}^{10} y_i A_i(q),$$

where x_i $(1 \leq i \leq 4)$ *and* y_i $(1 \leq i \leq 10)$ *are listed in* Table 3.

Proof. By (2.3), (2.5) and Theorem 2.1, we have $\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6) \in M_4(\Gamma_0(24), \chi_4)$. Appealing to Theorem 3.1(c) and using MAPLE, we obtain the asserted results. ■

Theorem 3.5. *Let* (i, j, k, l) *be any of the octonary quadratic forms listed in the second column of Table 1. Then*

$$\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6) = x_1 E_{4,\chi_0,\chi_5}(q) + x_2 E_{4,\chi_2,\chi_3}(q) + x_3 E_{4,\chi_3,\chi_2}(q) + x_4 E_{4,\chi_5,\chi_0}(q) + x_5 E_{4,\chi_0,\chi_5}(q^2) + x_6 E_{4,\chi_3,\chi_2}(q^2) + \sum_{i=1}^{8} y_i B_i(q),$$

where x_i $(1 \leq i \leq 6)$ *and* y_i $(1 \leq i \leq 8)$ *are listed in* Table 4.

Proof. By (2.3), (2.5) and Theorem 2.1, we have $\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6) \in M_4(\Gamma_0(24), \chi_5)$. Appealing to Theorem 3.2(c) and using MAPLE, we obtain the asserted results. ■

Theorem 3.6. *Let* (i, j, k, l) *be any of the octonary quadratic forms listed in the third column of Table 1. Then*

$$\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6) = x_1 E_{4,\chi_0,\chi_6}(q) + x_2 E_{4,\chi_1,\chi_3}(q) + x_3 E_{4,\chi_3,\chi_1}(q) + x_4 E_{4,\chi_6,\chi_0}(q)$$

$$+\sum_{i=1}^{10} y_i C_i(q),$$

where x_i $(1 \leq i \leq 4)$ *and* y_i $(1 \leq i \leq 10)$ *are listed in* Table 5.

Proof. By (2.3), (2.5) and Theorem 2.1, we have $\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6) \in M_4(\Gamma_0(24), \chi_6)$. Appealing to Theorem 3.3(c) and using MAPLE, we obtain the asserted results. ■

We now present formulas for $N(1^i, 2^j, 3^k, 6^l; n)$ for each of the octonary quadratic forms (i, j, k, l) listed in Table 1 in Theorems 3.7–3.9.

Theorem 3.7. *Let* $n \in \mathbb{N}$. *Let* (i, j, k, l) *be any of the octonary quadratic forms listed in the first column of Table 1. Then*

$$\begin{aligned} N(1^i, 2^j, 3^k, 6^l; n) &= x_1\sigma_{(3,\chi_0,\chi_4)}(n) + x_2\sigma_{(3,\chi_4,\chi_0)}(n) + x_3\sigma_{(3,\chi_0,\chi_4)}(n/3) + x_4\sigma_{(3,\chi_4,\chi_0)}(n/3) \\ &\quad + \sum_{i=1}^{10} y_i a_i(n), \end{aligned}$$

where x_i $(1 \leq i \leq 4)$ *and* y_i $(1 \leq i \leq 10)$ *are listed in* Table 3.

Proof. This follows from (2.2), (2.11), (2.18), (3.1) and Theorem 3.4. ■

Theorem 3.8. *Let* $n \in \mathbb{N}$. *Let* (i, j, k, l) *be any of the octonary quadratic forms listed in the second column of Table 1. Then*

$$\begin{aligned} N(1^i, 2^j, 3^k, 6^l; n) &= x_1\sigma_{(3,\chi_0,\chi_5)}(n) + x_2\sigma_{(3,\chi_2,\chi_3)}(n) + x_3\sigma_{(3,\chi_3,\chi_2)}(n) + x_4\sigma_{(3,\chi_5,\chi_0)}(n) \\ &\quad + x_5\sigma_{(3,\chi_0,\chi_5)}(n/2) + x_6\sigma_{(3,\chi_3,\chi_2)}(n/2) + \sum_{i=1}^{8} y_i b_i(n), \end{aligned}$$

where x_i $(1 \leq i \leq 6)$ *and* y_i $(1 \leq i \leq 8)$ *are listed in* Table 4.

Proof. The assertions follow from (2.2), (2.12), (2.15), (2.17), (2.19), (3.2) and Theorem 3.5. ■

Theorem 3.9. *Let* $n \in \mathbb{N}$. *Let* (i, j, k, l) *be any of the octonary quadratic forms listed in the third column of Table 1. Then*

$$\begin{aligned} N(1^i, 2^j, 3^k, 6^l; n) &= x_1\sigma_{(3,\chi_0,\chi_6)}(n) + x_2\sigma_{(3,\chi_1,\chi_3)}(n) + x_3\sigma_{(3,\chi_3,\chi_1)}(n) + x_4\sigma_{(3,\chi_6,\chi_0)}(n) \\ &\quad + \sum_{i=1}^{10} y_i c_i(n), \end{aligned}$$

where x_i $(1 \leq i \leq 4)$ *and* y_i $(1 \leq i \leq 10)$ *are listed in* Table 5.

Proof. The assertions follow from (2.2), (2.13), (2.14), (2.16), (2.20), (3.3) and Theorem 3.6. ■

TABLE 3. Values of x_i ($1 \le i \le 4$) and y_j ($1 \le j \le 10$) for Theorems 3.4 and 3.7

No.	(i,j,k,l)	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
1	$(0,2,1,5)$	$\frac{4}{451}$	$\frac{32}{451}$	$\frac{78}{451}$	$-\frac{624}{451}$	$-\frac{896}{451}$	$\frac{1544}{451}$	$\frac{720}{451}$	$\frac{2480}{451}$	$\frac{8752}{451}$	$\frac{12752}{451}$	$-\frac{3152}{451}$	$\frac{5312}{451}$	$\frac{860}{451}$	$-\frac{320}{41}$
2	$(0,2,3,3)$	$\frac{4}{451}$	$\frac{64}{451}$	$\frac{78}{451}$	$-\frac{1248}{451}$	$-\frac{1604}{451}$	$\frac{1288}{451}$	$\frac{2316}{451}$	$\frac{1572}{451}$	$\frac{18916}{451}$	$\frac{444}{11}$	$-\frac{7164}{451}$	$\frac{14112}{451}$	$\frac{1536}{451}$	$-\frac{6320}{451}$
3	$(0,2,5,1)$	$\frac{4}{451}$	$\frac{128}{451}$	$\frac{78}{451}$	$-\frac{2496}{451}$	$-\frac{1216}{451}$	$\frac{776}{451}$	$\frac{3704}{451}$	$-\frac{5656}{451}$	$\frac{19400}{451}$	$\frac{27304}{451}$	$-\frac{35032}{451}$	$\frac{46144}{451}$	$\frac{1084}{451}$	$-\frac{4704}{451}$
4	$(0,4,1,3)$	$-\frac{8}{451}$	$\frac{64}{451}$	$\frac{90}{451}$	$\frac{720}{451}$	$-\frac{1532}{451}$	$\frac{3104}{451}$	$-\frac{1780}{451}$	$\frac{14532}{451}$	$\frac{4948}{451}$	$\frac{7948}{451}$	$\frac{29780}{451}$	$-\frac{37088}{451}$	$\frac{36}{11}$	$-\frac{6128}{451}$
5	$(0,4,3,1)$	$-\frac{8}{451}$	$\frac{128}{451}$	$\frac{90}{451}$	$\frac{1440}{451}$	$-\frac{6392}{451}$	$\frac{2592}{451}$	$-\frac{2360}{451}$	$\frac{30264}{451}$	$\frac{27736}{451}$	$-\frac{11816}{451}$	$\frac{81944}{451}$	$-\frac{104768}{451}$	$\frac{6272}{451}$	$-\frac{25504}{451}$
6	$(0,6,1,1)$	$\frac{28}{451}$	$\frac{224}{451}$	$\frac{54}{451}$	$-\frac{432}{451}$	$\frac{9800}{451}$	$\frac{3592}{451}$	$-\frac{3816}{451}$	$-\frac{23640}{451}$	$-\frac{58456}{451}$	$\frac{37112}{451}$	$-\frac{84056}{451}$	$\frac{108032}{451}$	$-\frac{10052}{451}$	$\frac{39648}{451}$
7	$(1,1,0,6)$	$-\frac{2}{451}$	$\frac{16}{451}$	$\frac{84}{451}$	$\frac{672}{451}$	$-\frac{128}{41}$	$\frac{776}{451}$	$\frac{580}{451}$	$\frac{6708}{451}$	$\frac{12020}{451}$	$\frac{3996}{451}$	$\frac{18228}{451}$	$-\frac{17472}{451}$	$\frac{56}{11}$	$-\frac{512}{41}$
8	$(1,1,2,4)$	$-\frac{2}{451}$	$\frac{32}{451}$	$\frac{84}{451}$	$\frac{1344}{451}$	$-\frac{1680}{451}$	$\frac{648}{451}$	$\frac{1296}{451}$	$\frac{9616}{451}$	$\frac{16528}{451}$	$\frac{4016}{451}$	$\frac{22864}{451}$	$-\frac{26848}{451}$	$\frac{232}{41}$	$-\frac{6704}{451}$
9	$(1,1,4,2)$	$-\frac{2}{451}$	$\frac{64}{451}$	$\frac{84}{451}$	$\frac{2688}{451}$	$-\frac{2224}{451}$	$\frac{392}{451}$	$\frac{84}{41}$	$\frac{13628}{451}$	$\frac{16524}{451}$	$-\frac{1356}{451}$	$\frac{37548}{451}$	$-\frac{45600}{451}$	$\frac{3064}{451}$	$-\frac{8848}{451}$
10	$(1,1,6,0)$	$-\frac{2}{451}$	$\frac{128}{451}$	$\frac{84}{451}$	$\frac{5376}{451}$	$\frac{296}{451}$	$-\frac{120}{451}$	$-\frac{1624}{451}$	$\frac{12632}{451}$	$-\frac{6936}{451}$	$-\frac{2248}{41}$	$\frac{57896}{451}$	$-\frac{68672}{451}$	$\frac{480}{451}$	$-\frac{5920}{451}$
11	$(1,3,0,4)$	$\frac{10}{451}$	$\frac{80}{451}$	$\frac{72}{451}$	$-\frac{576}{451}$	$-\frac{2732}{451}$	$\frac{2056}{451}$	$\frac{4096}{451}$	$\frac{9480}{451}$	$\frac{34344}{451}$	$\frac{23352}{451}$	$\frac{4584}{451}$	$-\frac{5088}{451}$	$\frac{3544}{451}$	$-\frac{10768}{451}$
12	$(1,3,2,2)$	$\frac{10}{451}$	$\frac{160}{451}$	$\frac{72}{451}$	$-\frac{1152}{451}$	$-\frac{812}{451}$	$\frac{1416}{451}$	$\frac{4396}{451}$	$-\frac{252}{451}$	$\frac{2428}{41}$	68	$-\frac{16844}{451}$	$\frac{32000}{451}$	$\frac{1544}{451}$	$-\frac{3008}{451}$
13	$(1,3,4,0)$	$\frac{10}{451}$	$\frac{320}{451}$	$\frac{72}{451}$	$-\frac{2304}{451}$	$\frac{6636}{451}$	$\frac{136}{451}$	$\frac{3192}{451}$	$-\frac{35952}{451}$	$-\frac{19232}{451}$	$\frac{25456}{451}$	$-\frac{104800}{451}$	$\frac{135040}{451}$	$-\frac{6064}{451}$	$\frac{26944}{451}$
14	$(1,5,0,2)$	$-\frac{26}{451}$	$\frac{208}{451}$	$\frac{108}{451}$	$\frac{864}{451}$	$-\frac{5512}{451}$	$\frac{2872}{451}$	$\frac{1964}{451}$	$\frac{27180}{451}$	$\frac{45068}{451}$	$\frac{16196}{451}$	$\frac{68044}{451}$	$-\frac{67072}{451}$	$\frac{152}{11}$	$-\frac{22048}{451}$
15	$(1,5,2,0)$	$-\frac{26}{451}$	$\frac{416}{451}$	$\frac{108}{451}$	$\frac{1728}{451}$	$-\frac{9704}{451}$	$\frac{1208}{451}$	$-\frac{2504}{451}$	$\frac{38088}{451}$	$\frac{36104}{451}$	$-\frac{1784}{41}$	$\frac{89608}{451}$	$-\frac{136480}{451}$	$\frac{10216}{451}$	$-\frac{38608}{451}$
16	$(1,7,0,0)$	$\frac{2}{11}$	$\frac{16}{11}$	0	0	$\frac{12}{11}$	$\frac{24}{11}$	$-\frac{56}{11}$	$-\frac{96}{11}$	$-\frac{32}{11}$	$\frac{64}{11}$	-32	32	$-\frac{8}{11}$	$\frac{80}{11}$
17	$(2,0,1,5)$	$\frac{4}{451}$	$\frac{64}{451}$	$\frac{78}{451}$	$-\frac{1248}{451}$	$\frac{200}{451}$	$\frac{1288}{451}$	$\frac{512}{451}$	$\frac{3376}{451}$	$\frac{13504}{451}$	$\frac{224}{11}$	$\frac{1856}{451}$	$-\frac{320}{451}$	$\frac{1536}{451}$	$\frac{896}{451}$
18	$(2,0,3,3)$	$\frac{4}{451}$	$\frac{128}{451}$	$\frac{78}{451}$	$-\frac{2496}{451}$	$\frac{588}{451}$	$\frac{776}{451}$	$\frac{1900}{451}$	$\frac{3364}{451}$	$\frac{13988}{451}$	$\frac{11068}{451}$	$\frac{2852}{451}$	$\frac{2848}{451}$	$\frac{1084}{451}$	$\frac{2512}{451}$
19	$(2,0,5,1)$	$\frac{4}{451}$	$\frac{256}{451}$	$\frac{78}{451}$	$-\frac{4992}{451}$	$\frac{288}{41}$	$-\frac{248}{451}$	$\frac{2872}{451}$	$-\frac{9288}{451}$	$-\frac{4888}{451}$	$\frac{13032}{451}$	$-\frac{15000}{451}$	$\frac{576}{11}$	$-\frac{1624}{451}$	$\frac{12960}{451}$
20	$(2,2,1,3)$	$-\frac{8}{451}$	$\frac{128}{451}$	$\frac{90}{451}$	$\frac{1440}{451}$	$-\frac{4588}{451}$	$\frac{2592}{451}$	$\frac{3052}{451}$	$\frac{24852}{451}$	$\frac{43972}{451}$	$\frac{15244}{451}$	$\frac{54884}{451}$	$-\frac{61472}{451}$	$\frac{6272}{451}$	$-\frac{18288}{451}$

No.	(i,j,k,l)	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
21	$(2,2,3,1)$	$-\frac{8}{451}$	$\frac{256}{451}$	$\frac{90}{451}$	$\frac{2880}{451}$	$-\frac{5288}{451}$	$\frac{1568}{451}$	$\frac{8}{41}$	$\frac{29256}{451}$	$\frac{33624}{451}$	$-\frac{4440}{451}$	$\frac{74424}{451}$	$-\frac{95808}{451}$	$\frac{6844}{451}$	$-\frac{20960}{451}$
22	$(2,4,1,1)$	$\frac{28}{451}$	$\frac{448}{451}$	$\frac{54}{451}$	$-\frac{864}{451}$	$-\frac{240}{451}$	$\frac{1800}{451}$	$\frac{5224}{451}$	$-\frac{312}{451}$	$\frac{33848}{451}$	$\frac{1000}{11}$	$-\frac{18824}{451}$	$\frac{42368}{451}$	$\frac{1568}{451}$	$-\frac{288}{451}$
23	$(3,1,0,4)$	$\frac{10}{451}$	$\frac{160}{451}$	$\frac{72}{451}$	$-\frac{1152}{451}$	$-\frac{6224}{451}$	$\frac{5024}{451}$	$\frac{6200}{451}$	$\frac{26808}{451}$	$\frac{6200}{41}$	88	$\frac{39080}{451}$	$-\frac{40160}{451}$	$\frac{8760}{451}$	$-\frac{24656}{451}$
24	$(3,1,2,2)$	$\frac{10}{451}$	$\frac{320}{451}$	$\frac{72}{451}$	$-\frac{2304}{451}$	$-\frac{2384}{451}$	$\frac{3744}{451}$	$\frac{4996}{451}$	$\frac{12756}{451}$	$\frac{51124}{451}$	$\frac{41692}{451}$	$\frac{8852}{451}$	$\frac{5152}{451}$	$\frac{4760}{451}$	$-\frac{9136}{451}$
25	$(3,1,4,0)$	$\frac{10}{451}$	$\frac{640}{451}$	$\frac{72}{451}$	$-\frac{4608}{451}$	$\frac{8904}{451}$	$\frac{1184}{451}$	$\frac{784}{451}$	$-\frac{31584}{451}$	$-\frac{28128}{451}$	$\frac{11424}{451}$	$-\frac{82272}{451}$	$\frac{2688}{11}$	$-\frac{6848}{451}$	$\frac{29120}{451}$
26	$(3,3,0,2)$	$-\frac{26}{451}$	$\frac{416}{451}$	$\frac{108}{451}$	$\frac{1728}{451}$	$-\frac{7900}{451}$	$\frac{4816}{451}$	$\frac{6516}{451}$	$\frac{43500}{451}$	$\frac{84812}{451}$	$\frac{3628}{41}$	$\frac{95020}{451}$	$-\frac{93184}{451}$	$\frac{10216}{451}$	$-\frac{31392}{451}$
27	$(3,3,2,0)$	$-\frac{26}{451}$	$\frac{832}{451}$	$\frac{108}{451}$	$\frac{3456}{451}$	$-\frac{5460}{451}$	$\frac{1488}{451}$	$-\frac{384}{41}$	$\frac{27432}{451}$	$\frac{14568}{451}$	$-\frac{29928}{451}$	$\frac{71400}{451}$	$-\frac{116544}{451}$	$\frac{7360}{451}$	$-\frac{21216}{451}$
28	$(3,5,0,0)$	$\frac{2}{11}$	$\frac{32}{11}$	0	0	$-\frac{8}{11}$	$-\frac{16}{11}$	$-\frac{32}{11}$	$-\frac{144}{11}$	$\frac{160}{11}$	$\frac{304}{11}$	$-\frac{736}{11}$	$\frac{736}{11}$	$\frac{40}{11}$	$\frac{16}{11}$
29	$(4,0,1,3)$	$-\frac{8}{451}$	$\frac{256}{451}$	$\frac{90}{451}$	$\frac{2880}{451}$	$-\frac{10700}{451}$	$\frac{8784}{451}$	$\frac{500}{41}$	$\frac{45492}{451}$	$\frac{107588}{451}$	$\frac{58700}{451}$	$\frac{76228}{451}$	$-\frac{81376}{451}$	$\frac{14060}{451}$	$-\frac{42608}{451}$
30	$(4,0,3,1)$	$-\frac{8}{451}$	$\frac{512}{451}$	$\frac{90}{451}$	$\frac{5760}{451}$	$-\frac{936}{41}$	$\frac{6736}{451}$	$-\frac{2232}{451}$	$\frac{48888}{451}$	$\frac{88696}{451}$	$\frac{10312}{451}$	$\frac{59384}{451}$	$-\frac{77888}{451}$	$\frac{13400}{451}$	$-\frac{40736}{451}$
31	$(4,2,1,1)$	$\frac{28}{451}$	$\frac{896}{451}$	$\frac{54}{451}$	$-\frac{1728}{451}$	$-\frac{5888}{451}$	$\frac{5432}{451}$	$\frac{8872}{451}$	$\frac{24696}{451}$	$\frac{88568}{451}$	$\frac{70424}{451}$	$\frac{25048}{451}$	$-\frac{2368}{451}$	$\frac{8572}{451}$	$-\frac{22432}{451}$
32	$(5,1,0,2)$	$-\frac{26}{451}$	$\frac{832}{451}$	$\frac{108}{451}$	$\frac{3456}{451}$	$-\frac{21696}{451}$	$\frac{12312}{451}$	$\frac{1748}{41}$	$\frac{97788}{451}$	$\frac{207596}{451}$	$\frac{108980}{451}$	$\frac{192268}{451}$	$-\frac{188704}{451}$	$\frac{25400}{451}$	$-\frac{86160}{451}$
33	$(5,1,2,0)$	$-\frac{26}{451}$	$\frac{1664}{451}$	$\frac{108}{451}$	$\frac{6912}{451}$	$-\frac{5992}{451}$	$\frac{5656}{451}$	$-\frac{4056}{451}$	$\frac{27768}{451}$	$\frac{29224}{451}$	$-\frac{14456}{451}$	$\frac{63848}{451}$	$-\frac{105536}{451}$	$\frac{8864}{451}$	$-\frac{29728}{451}$
34	$(5,3,0,0)$	$\frac{2}{11}$	$\frac{64}{11}$	0	0	$-\frac{4}{11}$	$-\frac{8}{11}$	$-\frac{72}{11}$	$-\frac{240}{11}$	$\frac{192}{11}$	$\frac{432}{11}$	$-\frac{1152}{11}$	$\frac{1152}{11}$	$\frac{48}{11}$	$\frac{64}{11}$
35	$(6,0,1,1)$	$\frac{28}{451}$	$\frac{1792}{451}$	$\frac{54}{451}$	$-\frac{3456}{451}$	$-\frac{17184}{451}$	$\frac{12696}{451}$	$\frac{30600}{451}$	$\frac{4824}{41}$	$\frac{183576}{451}$	$\frac{18312}{41}$	$\frac{170520}{451}$	$-\frac{3648}{11}$	$\frac{20776}{451}$	$-\frac{66720}{451}$
36	$(7,1,0,0)$	$\frac{2}{11}$	$\frac{128}{11}$	0	0	$-\frac{40}{11}$	$-\frac{80}{11}$	$\frac{112}{11}$	$\frac{96}{11}$	$\frac{256}{11}$	$\frac{160}{11}$	$\frac{128}{11}$	$-\frac{128}{11}$	$\frac{64}{11}$	$-\frac{192}{11}$

TABLE 4. Values of x_i $(1 \le i \le 6)$ and y_j $(1 \le j \le 8)$ for Theorems 3.5 and 3.8

No.	(i,j,k,l)	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
1	$(0,1,2,5)$	0	$-\frac{1}{23}$	0	$\frac{1}{23}$	$\frac{1}{23}$	$\frac{1}{23}$	0	$\frac{28}{23}$	$\frac{64}{23}$	0	$-\frac{64}{23}$	$\frac{312}{23}$	$\frac{128}{23}$	$\frac{56}{23}$
2	$(0,1,4,3)$	0	$-\frac{2}{23}$	0	$\frac{2}{23}$	$\frac{1}{23}$	$\frac{1}{23}$	0	$\frac{12}{23}$	$\frac{128}{23}$	0	$-\frac{128}{23}$	$\frac{656}{23}$	$\frac{256}{23}$	$\frac{144}{23}$
3	$(0,1,6,1)$	0	$-\frac{4}{23}$	0	$\frac{4}{23}$	$\frac{1}{23}$	$\frac{1}{23}$	0	$-\frac{20}{23}$	$\frac{164}{23}$	0	$-\frac{440}{23}$	$\frac{976}{23}$	$\frac{328}{23}$	$\frac{320}{23}$
4	$(0,3,2,3)$	0	$\frac{1}{23}$	0	$\frac{3}{23}$	$\frac{1}{23}$	$-\frac{3}{23}$	$-\frac{4}{23}$	$\frac{124}{23}$	$\frac{12}{23}$	0	$\frac{304}{23}$	$-\frac{152}{23}$	$-\frac{240}{23}$	$-\frac{248}{23}$
5	$(0,3,4,1)$	0	$\frac{2}{23}$	0	$\frac{6}{23}$	$\frac{1}{23}$	$-\frac{3}{23}$	$-\frac{8}{23}$	$\frac{108}{23}$	$\frac{24}{23}$	$-\frac{272}{23}$	$\frac{608}{23}$	$-\frac{96}{23}$	$-\frac{480}{23}$	$-\frac{528}{23}$
6	$(0,5,2,1)$	0	$-\frac{1}{23}$	0	$\frac{9}{23}$	$\frac{1}{23}$	$\frac{9}{23}$	$-\frac{8}{23}$	$\frac{140}{23}$	$-\frac{152}{23}$	$\frac{384}{23}$	$-\frac{320}{23}$	$-\frac{744}{23}$	$\frac{640}{23}$	$\frac{280}{23}$
7	$(1,0,1,6)$	0	$\frac{1}{23}$	0	$\frac{1}{23}$	$\frac{1}{23}$	$-\frac{1}{23}$	$\frac{44}{23}$	0	$\frac{20}{23}$	8	0	$\frac{104}{23}$	0	0
8	$(1,0,3,4)$	0	$\frac{2}{23}$	0	$\frac{2}{23}$	$\frac{1}{23}$	$-\frac{1}{23}$	$\frac{42}{23}$	0	$\frac{86}{23}$	$\frac{232}{23}$	0	$\frac{328}{23}$	0	0
9	$(1,0,5,2)$	0	$\frac{4}{23}$	0	$\frac{4}{23}$	$\frac{1}{23}$	$-\frac{1}{23}$	$\frac{38}{23}$	0	$\frac{126}{23}$	$\frac{144}{23}$	0	$\frac{592}{23}$	0	0
10	$(1,0,7,0)$	$\frac{1}{23}$	$\frac{8}{23}$	$\frac{1}{23}$	$\frac{8}{23}$	0	0	$\frac{28}{23}$	0	$\frac{140}{23}$	$-\frac{224}{23}$	0	$\frac{672}{23}$	0	0
11	$(1,2,1,4)$	0	$-\frac{1}{23}$	0	$\frac{3}{23}$	$\frac{1}{23}$	$\frac{3}{23}$	$\frac{44}{23}$	$\frac{56}{23}$	$\frac{148}{23}$	$\frac{280}{23}$	$-\frac{128}{23}$	$\frac{600}{23}$	$\frac{256}{23}$	$\frac{112}{23}$
12	$(1,2,3,2)$	0	$-\frac{2}{23}$	0	$\frac{6}{23}$	$\frac{1}{23}$	$\frac{3}{23}$	$\frac{42}{23}$	$\frac{24}{23}$	$\frac{158}{23}$	$\frac{328}{23}$	$-\frac{256}{23}$	$\frac{776}{23}$	$\frac{512}{23}$	$\frac{288}{23}$
13	$(1,2,5,0)$	0	$-\frac{4}{23}$	0	$\frac{12}{23}$	$\frac{1}{23}$	$\frac{3}{23}$	$\frac{38}{23}$	$-\frac{40}{23}$	$\frac{86}{23}$	$\frac{240}{23}$	$-\frac{880}{23}$	$\frac{208}{23}$	$\frac{656}{23}$	$\frac{640}{23}$
14	$(1,4,1,2)$	0	$\frac{1}{23}$	0	$\frac{9}{23}$	$\frac{1}{23}$	$-\frac{9}{23}$	$\frac{36}{23}$	$\frac{128}{23}$	$\frac{172}{23}$	8	$\frac{480}{23}$	24	$-\frac{224}{23}$	$-\frac{256}{23}$
15	$(1,4,3,0)$	0	$\frac{2}{23}$	0	$\frac{18}{23}$	$\frac{1}{23}$	$-\frac{9}{23}$	$\frac{26}{23}$	$\frac{64}{23}$	$\frac{22}{23}$	$-\frac{312}{23}$	$\frac{224}{23}$	$\frac{104}{23}$	$-\frac{1184}{23}$	$-\frac{640}{23}$
16	$(1,6,1,0)$	0	$-\frac{1}{23}$	0	$\frac{27}{23}$	$\frac{1}{23}$	$\frac{27}{23}$	$\frac{20}{23}$	$\frac{24}{23}$	$-\frac{132}{23}$	$-\frac{40}{23}$	$-\frac{160}{23}$	$-\frac{360}{23}$	$\frac{1056}{23}$	$\frac{48}{23}$
17	$(2,1,0,5)$	0	$\frac{1}{23}$	0	$\frac{3}{23}$	$\frac{1}{23}$	$-\frac{3}{23}$	$\frac{88}{23}$	$\frac{124}{23}$	$\frac{104}{23}$	16	$-\frac{64}{23}$	$\frac{216}{23}$	$\frac{128}{23}$	$-\frac{248}{23}$
18	$(2,1,2,3)$	0	$\frac{2}{23}$	0	$\frac{6}{23}$	$\frac{1}{23}$	$-\frac{3}{23}$	$\frac{84}{23}$	$\frac{108}{23}$	$\frac{116}{23}$	$\frac{464}{23}$	$\frac{240}{23}$	$\frac{272}{23}$	$-\frac{112}{23}$	$-\frac{160}{23}$
19	$(2,1,4,1)$	0	$\frac{4}{23}$	0	$\frac{12}{23}$	$\frac{1}{23}$	$-\frac{3}{23}$	$\frac{76}{23}$	$\frac{76}{23}$	$\frac{48}{23}$	$\frac{288}{23}$	$\frac{296}{23}$	$\frac{16}{23}$	$-\frac{408}{23}$	$-\frac{352}{23}$

No.	(i,j,k,l)	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
20	$(2,3,0,3)$	0	$-\frac{1}{23}$	0	$\frac{9}{23}$	$\frac{1}{23}$	$\frac{9}{23}$	$\frac{84}{23}$	$\frac{140}{23}$	$\frac{308}{23}$	$\frac{752}{23}$	$\frac{48}{23}$	$\frac{1096}{23}$	$\frac{272}{23}$	$\frac{280}{23}$
21	$(2,3,2,1)$	0	$-\frac{2}{23}$	0	$\frac{18}{23}$	$\frac{1}{23}$	$\frac{9}{23}$	$\frac{76}{23}$	$\frac{60}{23}$	$\frac{156}{23}$	$\frac{576}{23}$	$-\frac{272}{23}$	$\frac{768}{23}$	$\frac{912}{23}$	$\frac{352}{23}$
22	$(2,5,0,1)$	0	$\frac{1}{23}$	0	$\frac{27}{23}$	$\frac{1}{23}$	$-\frac{27}{23}$	$\frac{64}{23}$	$\frac{140}{23}$	$\frac{192}{23}$	16	$\frac{640}{23}$	$\frac{824}{23}$	$\frac{192}{23}$	$-\frac{280}{23}$
23	$(3,0,1,4)$	0	$-\frac{2}{23}$	0	$\frac{6}{23}$	$\frac{1}{23}$	$\frac{3}{23}$	$\frac{134}{23}$	$\frac{208}{23}$	$\frac{66}{23}$	$\frac{696}{23}$	$\frac{112}{23}$	$\frac{40}{23}$	$\frac{144}{23}$	$-\frac{80}{23}$
24	$(3,0,3,2)$	0	$-\frac{4}{23}$	0	$\frac{12}{23}$	$\frac{1}{23}$	$\frac{3}{23}$	$\frac{130}{23}$	$\frac{144}{23}$	$-\frac{6}{23}$	$\frac{976}{23}$	$\frac{224}{23}$	$-\frac{528}{23}$	$\frac{288}{23}$	$-\frac{96}{23}$
25	$(3,0,5,0)$	$\frac{1}{23}$	$-\frac{8}{23}$	$-\frac{3}{23}$	$\frac{24}{23}$	0	0	$\frac{124}{23}$	$\frac{16}{23}$	$-\frac{324}{23}$	$\frac{992}{23}$	$-\frac{160}{23}$	$-\frac{2912}{23}$	$\frac{96}{23}$	$-\frac{128}{23}$
26	$(3,2,1,2)$	0	$\frac{2}{23}$	0	$\frac{18}{23}$	$\frac{1}{23}$	$-\frac{9}{23}$	$\frac{118}{23}$	$\frac{248}{23}$	$\frac{298}{23}$	$\frac{792}{23}$	$\frac{592}{23}$	$\frac{840}{23}$	$-\frac{80}{23}$	$-\frac{272}{23}$
27	$(3,2,3,0)$	0	$\frac{4}{23}$	0	$\frac{36}{23}$	$\frac{1}{23}$	$-\frac{9}{23}$	$\frac{98}{23}$	$\frac{120}{23}$	$-\frac{94}{23}$	$-\frac{16}{23}$	$\frac{80}{23}$	$-\frac{976}{23}$	$-\frac{1264}{23}$	$-\frac{672}{23}$
28	$(3,4,1,0)$	0	$-\frac{2}{23}$	0	$\frac{54}{23}$	$\frac{1}{23}$	$\frac{27}{23}$	$\frac{86}{23}$	$-\frac{16}{23}$	$-\frac{126}{23}$	$\frac{216}{23}$	$-\frac{688}{23}$	$\frac{8}{23}$	$\frac{1008}{23}$	$\frac{176}{23}$
29	$(4,1,0,3)$	0	$-\frac{2}{23}$	0	$\frac{18}{23}$	$\frac{1}{23}$	$\frac{9}{23}$	$\frac{168}{23}$	$\frac{428}{23}$	$\frac{616}{23}$	$\frac{1312}{23}$	$\frac{96}{23}$	$\frac{1872}{23}$	$\frac{544}{23}$	$-\frac{16}{23}$
30	$(4,1,2,1)$	0	$-\frac{4}{23}$	0	$\frac{36}{23}$	$\frac{1}{23}$	$\frac{9}{23}$	$\frac{152}{23}$	$\frac{268}{23}$	$\frac{220}{23}$	$\frac{960}{23}$	$\frac{8}{23}$	$\frac{848}{23}$	$\frac{904}{23}$	$\frac{128}{23}$
31	$(4,3,0,1)$	0	$\frac{2}{23}$	0	$\frac{54}{23}$	$\frac{1}{23}$	$-\frac{27}{23}$	$\frac{128}{23}$	$\frac{300}{23}$	$\frac{384}{23}$	$\frac{1040}{23}$	$\frac{1280}{23}$	$\frac{1440}{23}$	$\frac{384}{23}$	$-\frac{240}{23}$
32	$(5,0,1,2)$	0	$\frac{4}{23}$	0	$\frac{36}{23}$	$\frac{1}{23}$	$-\frac{9}{23}$	$\frac{190}{23}$	$\frac{672}{23}$	$\frac{918}{23}$	$\frac{720}{23}$	$\frac{448}{23}$	$\frac{4176}{23}$	$\frac{576}{23}$	$\frac{64}{23}$
33	$(5,0,3,0)$	$\frac{1}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	$\frac{72}{23}$	0	0	$\frac{140}{23}$	$\frac{416}{23}$	$\frac{284}{23}$	$-\frac{1120}{23}$	$-\frac{832}{23}$	$\frac{1696}{23}$	$-\frac{832}{23}$	0
34	$(5,2,1,0)$	0	$-\frac{4}{23}$	0	$\frac{108}{23}$	$\frac{1}{23}$	$\frac{27}{23}$	$\frac{126}{23}$	$\frac{88}{23}$	$-\frac{114}{23}$	$\frac{176}{23}$	$-\frac{1008}{23}$	$-\frac{176}{23}$	$\frac{912}{23}$	$\frac{64}{23}$
35	$(6,1,0,1)$	0	$\frac{4}{23}$	0	$\frac{108}{23}$	$\frac{1}{23}$	$-\frac{27}{23}$	$\frac{164}{23}$	$\frac{620}{23}$	$\frac{1320}{23}$	$\frac{2016}{23}$	$\frac{1640}{23}$	$\frac{4880}{23}$	$\frac{1320}{23}$	$-\frac{160}{23}$
36	$(7,0,1,0)$	$\frac{1}{23}$	$-\frac{8}{23}$	$-\frac{27}{23}$	$\frac{216}{23}$	0	0	$\frac{140}{23}$	$\frac{112}{23}$	$-\frac{84}{23}$	$\frac{1120}{23}$	$-\frac{1120}{23}$	$-\frac{2912}{23}$	$\frac{672}{23}$	$-\frac{896}{23}$

Table 5. Values of x_i $(1 \le i \le 4)$ and y_j $(1 \le j \le 10)$ for Theorems 3.6 and 3.9

No.	(i,j,k,l)	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
1	$(0,1,1,6)$	$\frac{1}{261}$	$-\frac{8}{261}$	$-\frac{1}{261}$	$\frac{8}{261}$	$\frac{392}{261}$	$\frac{88}{29}$	$-\frac{148}{87}$	$-\frac{3136}{261}$	$-\frac{136}{29}$	$\frac{9472}{261}$	$\frac{2116}{261}$	$-\frac{3136}{261}$	$-\frac{392}{261}$	$-\frac{400}{261}$
2	$(0,1,3,4)$	$\frac{1}{261}$	$-\frac{16}{261}$	$-\frac{1}{261}$	$\frac{16}{261}$	$\frac{304}{87}$	$\frac{132}{29}$	$-\frac{72}{29}$	$-\frac{2432}{87}$	$-\frac{272}{29}$	$\frac{5600}{87}$	$\frac{1256}{87}$	$-\frac{2432}{87}$	$-\frac{304}{87}$	$-\frac{308}{87}$
3	$(0,1,5,2)$	$\frac{1}{261}$	$-\frac{32}{261}$	$-\frac{1}{261}$	$\frac{32}{261}$	$\frac{1952}{261}$	$\frac{220}{29}$	$-\frac{700}{87}$	$-\frac{15616}{261}$	$-\frac{776}{29}$	$\frac{31456}{261}$	$\frac{10204}{261}$	$-\frac{15616}{261}$	$-\frac{1952}{261}$	$-\frac{68}{9}$
4	$(0,1,7,0)$	$\frac{1}{261}$	$-\frac{64}{261}$	$-\frac{1}{261}$	$\frac{64}{261}$	$\frac{448}{29}$	$\frac{280}{29}$	$-\frac{672}{29}$	$-\frac{3584}{29}$	$-\frac{2016}{29}$	$\frac{5824}{29}$	$\frac{2912}{29}$	$-\frac{3584}{29}$	$-\frac{448}{29}$	$-\frac{336}{29}$
5	$(0,3,1,4)$	$\frac{1}{261}$	$\frac{8}{261}$	$\frac{1}{87}$	$\frac{8}{87}$	$-\frac{104}{29}$	$\frac{60}{29}$	$\frac{176}{29}$	32	$\frac{672}{29}$	$-\frac{224}{29}$	$-\frac{848}{29}$	$\frac{1280}{29}$	$\frac{100}{29}$	$\frac{100}{29}$
6	$(0,3,3,2)$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{1}{87}$	$\frac{16}{87}$	$-\frac{1000}{261}$	$\frac{128}{87}$	$\frac{2036}{261}$	$\frac{736}{29}$	$\frac{3368}{87}$	$-\frac{5536}{261}$	$-\frac{11500}{261}$	$\frac{24064}{261}$	$\frac{932}{261}$	$\frac{32}{9}$
7	$(0,3,5,0)$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{1}{87}$	$\frac{32}{87}$	$-\frac{376}{87}$	$\frac{8}{29}$	$\frac{632}{87}$	$\frac{352}{29}$	$\frac{1328}{29}$	$-\frac{4192}{87}$	$-\frac{9544}{87}$	$\frac{16384}{87}$	$\frac{332}{87}$	$\frac{328}{87}$
8	$(0,5,1,2)$	$\frac{1}{261}$	$-\frac{8}{261}$	$-\frac{1}{29}$	$\frac{8}{29}$	$-\frac{1384}{261}$	$\frac{224}{87}$	$\frac{1484}{261}$	$\frac{5696}{87}$	$\frac{152}{87}$	$\frac{1664}{261}$	$\frac{1964}{261}$	$-\frac{15680}{261}$	$\frac{1328}{261}$	$\frac{1288}{261}$
9	$(0,5,3,0)$	$\frac{1}{261}$	$-\frac{16}{261}$	$-\frac{1}{29}$	$\frac{16}{29}$	$-\frac{1376}{87}$	$-\frac{292}{29}$	$\frac{1984}{87}$	$\frac{3840}{29}$	$\frac{544}{29}$	$-\frac{16736}{87}$	$-\frac{1664}{87}$	$-\frac{12160}{87}$	$\frac{1336}{87}$	$\frac{1316}{87}$
10	$(0,7,1,0)$	$\frac{1}{261}$	$\frac{8}{261}$	$\frac{3}{29}$	$\frac{24}{29}$	$\frac{168}{29}$	$\frac{420}{29}$	$-\frac{1064}{29}$	$-\frac{672}{29}$	$-\frac{1680}{29}$	$\frac{5600}{29}$	$\frac{2744}{29}$	$\frac{1792}{29}$	$-\frac{196}{29}$	$-\frac{196}{29}$
11	$(1,0,0,7)$	$\frac{1}{261}$	$\frac{8}{261}$	$\frac{1}{261}$	$\frac{8}{261}$	$\frac{28}{29}$	$-\frac{28}{29}$	$\frac{28}{29}$	$\frac{56}{29}$	$\frac{56}{29}$	$-\frac{168}{29}$	$-\frac{84}{29}$	0	$\frac{28}{29}$	$\frac{28}{29}$
12	$(1,0,2,5)$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{1004}{261}$	$\frac{172}{87}$	$-\frac{16}{9}$	$-\frac{4544}{261}$	$-\frac{344}{87}$	$\frac{2560}{87}$	$\frac{568}{87}$	0	$-\frac{172}{87}$	$-\frac{172}{87}$
13	$(1,0,4,3)$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{488}{87}$	$\frac{112}{29}$	$-\frac{284}{87}$	$-\frac{2792}{87}$	$-\frac{224}{29}$	$\frac{1736}{29}$	$\frac{388}{29}$	0	$-\frac{112}{29}$	$-\frac{112}{29}$
14	$(1,0,6,1)$	$\frac{1}{261}$	$\frac{64}{261}$	$\frac{1}{261}$	$\frac{64}{261}$	$\frac{2384}{261}$	$\frac{664}{87}$	$-\frac{2672}{261}$	$-\frac{18128}{261}$	$-\frac{1328}{87}$	$\frac{9808}{87}$	$\frac{1312}{87}$	0	$-\frac{664}{87}$	$-\frac{664}{87}$
15	$(1,2,0,5)$	$\frac{1}{261}$	$-\frac{8}{261}$	$-\frac{1}{87}$	$\frac{8}{87}$	$\frac{2036}{261}$	$\frac{776}{87}$	$-\frac{1528}{261}$	$-\frac{4232}{87}$	$-\frac{1312}{87}$	$\frac{26312}{261}$	$\frac{6776}{261}$	$-\frac{6272}{261}$	$-\frac{1528}{261}$	$-\frac{1544}{261}$
16	$(1,2,2,3)$	$\frac{1}{261}$	$-\frac{16}{261}$	$-\frac{1}{87}$	$\frac{16}{87}$	$\frac{20}{3}$	$\frac{200}{29}$	$-\frac{188}{87}$	$-\frac{1040}{29}$	$-\frac{416}{29}$	$\frac{8368}{87}$	$\frac{2092}{87}$	$-\frac{4864}{87}$	$-\frac{416}{87}$	$-\frac{424}{87}$
17	$(1,2,4,1)$	$\frac{1}{261}$	$-\frac{32}{261}$	$-\frac{1}{87}$	$\frac{32}{87}$	$\frac{2192}{261}$	$\frac{596}{87}$	$-\frac{1768}{261}$	$-\frac{4376}{87}$	$-\frac{3208}{87}$	$\frac{28952}{261}$	$\frac{14672}{261}$	$-\frac{31232}{261}$	$-\frac{1732}{261}$	$-\frac{1772}{261}$
18	$(1,4,0,3)$	$\frac{1}{261}$	$\frac{8}{261}$	$\frac{1}{29}$	$\frac{8}{29}$	$-\frac{36}{29}$	$\frac{92}{29}$	$\frac{388}{29}$	$\frac{760}{29}$	$\frac{1128}{29}$	$\frac{536}{29}$	$-\frac{1516}{29}$	$\frac{1408}{29}$	$\frac{84}{29}$	$\frac{84}{29}$
19	$(1,4,2,1)$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{1}{29}$	$\frac{16}{29}$	$\frac{1340}{261}$	$\frac{692}{87}$	$-\frac{904}{261}$	$-\frac{3328}{87}$	$\frac{1976}{87}$	$\frac{21632}{261}$	$-\frac{3088}{261}$	$\frac{29440}{261}$	$-\frac{988}{261}$	$-\frac{1004}{261}$
20	$(1,6,0,1)$	$\frac{1}{261}$	$-\frac{8}{261}$	$-\frac{3}{29}$	$\frac{24}{29}$	$\frac{884}{261}$	$\frac{656}{87}$	$\frac{80}{261}$	$-\frac{712}{87}$	$\frac{368}{87}$	$\frac{19592}{261}$	$\frac{10496}{261}$	$-\frac{10496}{261}$	$-\frac{544}{261}$	$-\frac{656}{261}$

No.	(i,j,k,l)	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
21	$(2,1,1,4)$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{1}{87}$	$\frac{16}{87}$	$\frac{1088}{261}$	$\frac{476}{87}$	$\frac{992}{261}$	$-\frac{192}{29}$	$\frac{1280}{87}$	$\frac{11168}{261}$	$-\frac{4192}{261}$	$\frac{7360}{261}$	$-\frac{112}{261}$	$-\frac{4}{9}$
22	$(2,1,3,2)$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{1}{87}$	$\frac{32}{87}$	$\frac{320}{87}$	$\frac{124}{29}$	$\frac{284}{87}$	$-\frac{112}{29}$	$\frac{632}{29}$	$\frac{2768}{87}$	$-\frac{1540}{87}$	$\frac{5248}{87}$	$-\frac{16}{87}$	$-\frac{20}{87}$
23	$(2,1,5,0)$	$\frac{1}{261}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{64}{87}$	$\frac{704}{261}$	$-\frac{184}{87}$	$-\frac{472}{261}$	$-\frac{416}{29}$	$\frac{1040}{87}$	$-\frac{9952}{261}$	$-\frac{14872}{261}$	$\frac{32512}{261}$	$\frac{80}{261}$	$\frac{1096}{261}$
24	$(2,3,1,2)$	$\frac{1}{261}$	$-\frac{16}{261}$	$-\frac{1}{29}$	$\frac{16}{29}$	$\frac{712}{87}$	$\frac{288}{29}$	$\frac{244}{87}$	$-\frac{800}{29}$	$-\frac{152}{29}$	$\frac{11104}{87}$	$\frac{2164}{87}$	$-\frac{6592}{87}$	$-\frac{404}{87}$	$-\frac{424}{87}$
25	$(2,3,3,0)$	$\frac{1}{261}$	$-\frac{32}{261}$	$-\frac{1}{29}$	$\frac{32}{29}$	$\frac{824}{261}$	$\frac{56}{87}$	$\frac{272}{261}$	$-\frac{496}{87}$	$-\frac{3760}{87}$	$\frac{560}{261}$	$\frac{4064}{261}$	$-\frac{44672}{261}$	$-\frac{28}{261}$	$-\frac{128}{261}$
26	$(2,5,1,0)$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{3}{29}$	$\frac{48}{29}$	$\frac{2096}{261}$	$\frac{644}{87}$	$-\frac{4504}{261}$	$-\frac{6016}{87}$	$-\frac{112}{87}$	$\frac{19616}{261}$	$\frac{10664}{261}$	$\frac{28864}{261}$	$-\frac{1528}{261}$	$-\frac{1580}{261}$
27	$(3,0,0,5)$	$\frac{1}{261}$	$-\frac{16}{261}$	$-\frac{1}{87}$	$\frac{16}{87}$	$\frac{20}{3}$	$\frac{316}{29}$	$\frac{160}{87}$	$-\frac{576}{29}$	$\frac{280}{29}$	$\frac{6976}{87}$	$-\frac{344}{87}$	$\frac{704}{87}$	$-\frac{68}{87}$	$-\frac{76}{87}$
28	$(3,0,2,3)$	$\frac{1}{261}$	$-\frac{32}{261}$	$-\frac{1}{87}$	$\frac{32}{87}$	$\frac{104}{261}$	$\frac{248}{87}$	$\frac{3452}{261}$	$\frac{3976}{87}$	$\frac{3056}{87}$	$-\frac{4456}{261}$	$-\frac{9340}{261}$	$\frac{2176}{261}$	$\frac{1400}{261}$	$\frac{1360}{261}$
29	$(3,0,4,1)$	$\frac{1}{261}$	$-\frac{64}{261}$	$-\frac{1}{87}$	$\frac{64}{87}$	$-\frac{352}{29}$	$-\frac{384}{29}$	32	$\frac{4432}{29}$	$\frac{1104}{29}$	$-\frac{6832}{29}$	$-\frac{1840}{29}$	$\frac{256}{29}$	$\frac{512}{29}$	$\frac{504}{29}$
30	$(3,2,0,3)$	$\frac{1}{261}$	$\frac{16}{261}$	$\frac{1}{29}$	$\frac{16}{29}$	$\frac{1340}{261}$	$\frac{1040}{87}$	$\frac{4316}{261}$	$\frac{848}{87}$	$\frac{4064}{87}$	$\frac{25808}{261}$	$-\frac{14572}{261}$	$\frac{12736}{261}$	$\frac{56}{261}$	$\frac{40}{261}$
31	$(3,2,2,1)$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{1}{29}$	$\frac{32}{29}$	$\frac{512}{87}$	$\frac{276}{29}$	$\frac{248}{87}$	$-\frac{440}{29}$	$\frac{1112}{29}$	$\frac{6584}{87}$	$-\frac{1648}{87}$	$\frac{9856}{87}$	$-\frac{100}{87}$	$-\frac{4}{3}$
32	$(3,4,0,1)$	$\frac{1}{261}$	$-\frac{16}{261}$	$-\frac{3}{29}$	$\frac{48}{29}$	$\frac{1108}{87}$	$\frac{436}{29}$	$-\frac{200}{87}$	$-\frac{1472}{29}$	$-\frac{56}{29}$	$\frac{17920}{87}$	$\frac{6208}{87}$	$-\frac{6208}{87}$	$-\frac{716}{87}$	$-\frac{772}{87}$
33	$(4,1,1,2)$	$\frac{1}{261}$	$-\frac{32}{261}$	$-\frac{1}{29}$	$\frac{32}{29}$	$\frac{2912}{261}$	$\frac{1796}{87}$	$\frac{3404}{261}$	$-\frac{1888}{87}$	$\frac{2504}{87}$	$\frac{54848}{261}$	$-\frac{1156}{261}$	$-\frac{11264}{261}$	$-\frac{1072}{261}$	$-\frac{1172}{261}$
34	$(4,1,3,0)$	$\frac{1}{261}$	$-\frac{64}{261}$	$-\frac{1}{29}$	$\frac{64}{29}$	$\frac{32}{29}$	$-\frac{56}{29}$	$\frac{160}{29}$	$\frac{640}{29}$	$-\frac{672}{29}$	$-\frac{2112}{29}$	$-\frac{1248}{29}$	$-\frac{3072}{29}$	$\frac{144}{29}$	$\frac{240}{29}$
35	$(4,3,1,0)$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{3}{29}$	$\frac{96}{29}$	$\frac{392}{87}$	$\frac{152}{29}$	$-\frac{904}{87}$	$-\frac{960}{29}$	16	$-\frac{64}{87}$	$-\frac{8}{3}$	$\frac{12544}{87}$	$-\frac{4}{87}$	$-\frac{56}{87}$
36	$(5,0,0,3)$	$\frac{1}{261}$	$\frac{32}{261}$	$\frac{1}{29}$	$\frac{32}{29}$	$\frac{2600}{87}$	$\frac{1552}{29}$	$\frac{596}{87}$	$-\frac{5080}{29}$	$\frac{416}{29}$	$\frac{51128}{87}$	$\frac{92}{87}$	$-\frac{1280}{87}$	$-\frac{1840}{87}$	$-\frac{64}{3}$
37	$(5,0,2,1)$	$\frac{1}{261}$	$\frac{64}{261}$	$\frac{1}{29}$	$\frac{64}{29}$	$\frac{12368}{261}$	$\frac{5624}{87}$	$-\frac{14752}{261}$	$-\frac{31408}{87}$	$-\frac{208}{87}$	$\frac{206000}{261}$	$\frac{24752}{261}$	$-\frac{3584}{261}$	$-\frac{10408}{261}$	$-\frac{10520}{261}$
38	$(5,2,0,1)$	$\frac{1}{261}$	$-\frac{32}{261}$	$-\frac{3}{29}$	$\frac{96}{29}$	$\frac{5072}{261}$	$\frac{2612}{87}$	$\frac{2216}{261}$	$-\frac{5560}{87}$	$\frac{2936}{87}$	$\frac{99128}{261}$	$\frac{18176}{261}$	$-\frac{18176}{261}$	$-\frac{3268}{261}$	$-\frac{3548}{261}$
39	$(6,1,1,0)$	$\frac{1}{261}$	$\frac{64}{261}$	$\frac{3}{29}$	$\frac{192}{29}$	$\frac{5600}{261}$	$\frac{2168}{87}$	$-\frac{7480}{261}$	$-\frac{16096}{87}$	$-\frac{3952}{87}$	$\frac{68768}{261}$	$\frac{9992}{261}$	$\frac{21760}{261}$	$-\frac{4288}{261}$	$-\frac{3608}{261}$
40	$(7,0,0,1)$	$\frac{1}{261}$	$-\frac{64}{261}$	$-\frac{3}{29}$	$\frac{192}{29}$	$-\frac{672}{29}$	0	$\frac{4816}{29}$	$\frac{11536}{29}$	$\frac{2352}{29}$	$-\frac{2800}{29}$	$-\frac{1792}{29}$	$\frac{1792}{29}$	$\frac{896}{29}$	$\frac{840}{29}$

Acknowledgments. The research was carried out during the first author's visit to Recep Tayyip Erdoğan University, Rize, Turkey. She would like to thank Recep Tayyip Erdoğan University for the hospitality during her stay. She thanks the Scientific and Technological Research Council of Turkey (TÜBİTAK) for their partial financial support. The research of the first author was also partially supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (RGPIN-418029-2013).

References

[1] A. Alaca, Ş. Alaca and K. S. Williams, *Fourteen octonary quadratic forms,* Int. J. Number Theory **6** (2010), 37-50.
[2] Ş. Alaca and Y. Kesicioğlu, *Representations by certain octonary quadratic forms whose coefficients are* 1*,* 2*,* 3 *and* 6*,* Int. J. Number Theory **10** (2014), 133-150.
[3] Ş. Alaca and Y. Kesicioğlu, *Representations by Certain Octonary Quadratic forms with Coefficients* 1*,* 2*,* 3 *and* 6*,* Integers, **15** (2015), 1-9.
[4] Ş. Alaca and K. S. Williams, *The number of representations of a positive integer by certain octonary quadratic forms,* Functiones et Approximatio **43** (2010), 45-54.
[5] B. Gordon and D. Sinor, *Multiplicative properties of η-products,* Lecture Notes in Math, vol.1395 Springer-Verlag, New York (1989), 173-200.
[6] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum,* Bornträger, Regiomonti, 1829. (Gesammelte Werke, Erster Band, Chelsea Publishing Co., New York, 1969, pp. 49-239.)
[7] L. J. P. Kilford, *Modular Forms, A classical and computational introduction,* Imperial College Press, London, 2008.
[8] G. Köhler, *Eta Products and Theta Series Identities,* Springer, 2011.
[9] B. Köklüce, *The representation numbers of three octonary quadratic forms,* Int. J. Number Theory **9** (2013), 505-516.
[10] G. Ligozat, *Courbes modulaires de genre 1,* Bull. Soc. Math. France **43** (1975), 5-80.
[11] B. Ramakrishnan and B. Sahu, *Evaluation of the convolution sums* $\sum_{l+15m=n}\sigma(l)\sigma(m)$ *and* $\sum_{3l+5m=n}\sigma(l)\sigma(m)$ *and an application,* Int. J. Number Theory **9** (2013), 799-809.
[12] J. P. Serre, *Modular forms of weight one and Galois representations,* Algebraic number fields: L -functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, 193-268.
[13] W. Stein, *Modular Forms, A Computational Approach,* Ameican Mathematical Society, 2007.

Ayşe Alaca
School of Mathematics and Statistics
Carleton University, Ottawa
Ontario, K1S 5B6, Canada
aalaca@math.carleton.ca

M. Nesibe Kesicioğlu
Department of Mathematics
Recep Tayyip Erdoğan University
Rize, 53100, Turkey
m.nesibe@gmail.com