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The Arens Triadjoints of some Bilinear Maps

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Abstract. In this paper we study the Arens triadjoints of some bilinear maps on vector lattices. In particular, we prove that, for Archimedean vector lattices A and B, the Arens triadjoint ***i)*** $T^{***} : A'' \times A'' \to B''$ of a positive orthosymmetric bilinear map $T : A \times A \to B$ is positive orthosymmetric, and ***ii)*** $T^{***} : A'' \times A'' \to A''$ of a bi-orthomorphism $T : A \times A \to A$ is a bi-orthomorphism. These generalize results on the order bidual of f-algebras and almost f-algebras in [4].

1. Introduction and Preliminaries

The Arens multiplication introduced in [3] on the bidual of various lattice ordered (or Riesz) algebras has been well documented (see, e.g., [4]). The more general question about Arens triadjoints of bilinear maps on products of vector lattices has recently aroused considerable interest (see, e.g., [7]). In Theorem 2.1 in [7] several properties of the Arens triadjoint maps are collected. For example, the adjoint of a bilinear map of order bounded variation is of order bounded variation and the triadjoint of such a map is separately order continuous. In this direction, as the extensions of the notions of classes of *almost f-algebras* [6] (a lattice ordered algebra A for which $a \wedge b = 0$ in A implies $ab = 0$) and *f-algebras* [5] (a lattice ordered algebra A with the property that $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$), we study the Arens triadjoints of some classes of bilinear maps on vector lattices (or Riesz spaces); mainly, orthosymmetric bilinear maps and bi-orthomorphisms: Let A and B be vector lattices. A bilinear map $T : A \times A \to B$ is said to be

(1) *orthosymmetric* if $x \wedge y = 0$ implies $T(x, y) = 0$ for all $x, y \in A$ (first appeared in a paper by G. Buskes and A. van Rooij in [10] in 2000).

(2) a *bi-orthomorphism* if it is a separately order bounded bilinear map such that $x \wedge y = 0$ in A implies $T(z, x) \wedge y = 0$ for all $z \in A^+$, when $A = B$ (first appears a paper by G. Buskes, R. Page, Jr. and R. Yilmaz in [11] in 2009).

It is obvious that every bi-orthomorphism is orthosymmetric. The class of orthosymmetric bilinear maps was introduced in [10] by G. Buskes and A. van Rooij. Subsequent developments have been made as a result of contributions by the same authors [9], G. Buskes and A. G. Kusraev [8], and M. A. Toumi [14]. In [14] it is proved that if A and B are Archimedean vector lattices, $(A')'_n, (B')'_n$ are their respective order continuous biduals and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then the triadjoint $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ of T is a positive orthosymmetric bilinear map. In particular, in Section 2 we extend this result to the whole $A'' \times A''$; that is, if A and B are Archimedean vector lattices, A'' and B'' are

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the order biduals of A and B respectively, then $T^{***} : A'' \times A'' \to B''$ is a positive orthosymmetric bilinear map whenever $T : A \times A \to B$ is so. It can be obtained similar results for the class of the Arens triadjoint of bi-orthomorphisms when $A = B$. So, all the results on the bidual of an f-algebra in the paper [4] can be reformulated and in Section 3 we obtain the following result: Let A be a vector lattice and $T : A \times A \to A$ be a bi-orthomorphism. Then the bilinear map $T^{***} : A'' \times A'' \to A''$ is a bi-orthomorphism. In other words, if $T \in Orth(A, A)$, then $T^{***} \in Orth(A'', A'')$, where $Orth(A, A)$ denotes the space of all bi-orthomorphisms on A.

R. Yilmaz and K. Rowlands in [15] in 2006 were the first to study bi-orthomorphisms what they called quasi-orthomorphisms. The notion of bi-orthomorphism, as given here, first appears in a paper by G. Buskes, R. Page, Jr. and R. Yilmaz in [11] in 2009. where it is proved that the space $Orth(A, A)$ of bi-orthomorphisms is a vector lattice, which contains the space $Orth(A)$ of orthomorphisms as a vector sublattice in case that A is a semiprime f-algebra. Moreover $Orth(A)$ is an ideal in the vector lattice $Orth(A, A)$ in case A is a semiprime Dedekind complete f-algebra. When exactly $Orth(A, A)$ is an f-algebra is indeed unclear. However, if A is an f-algebra which is also a Banach algebra with a minimal ultra-approximate identity, then $Orth(A, A)$ is a Banach f-algebra (see Theorem 4.6 in [15] and Proposition 3.14 in [11]).

From here on, let A, B, and C be Archimedean vector lattices and A', B', C' be their respective duals.

A bilinear map $T : A \times B \to C$ can be extended in a natural way to the bilinear map $T^{***} : A'' \times B'' \to C''$ constructed in the following stages:

$$\begin{array}{ll} T^{*} : C' \times A \to B', & T^{*}(f, x)(y) = f(T(x, y)) \\ T^{**} : B'' \times C' \to A', & T^{**}(G, f)(x) = G(T^{*}(f, x)) \\ T^{***} : A'' \times B'' \to C'', & T^{***}(F, G)(f) = F(T^{**}(G, f)) \end{array}$$

for all $x \in A, y \in B, f \in C', F \in A'', G \in B''$ (so-called the *first Arens adjoint* of T).

Another extension of a bilinear map $T : A \times B \to C$ is the map ${}^{***}T : A'' \times B'' \to C''$ constructed in the following stages:

$$\begin{array}{ll} {}^{*}T : B \times C' \to A', & {}^{*}T(y, f)(x) = f(T(x, y)) \\ {}^{**}T : C' \times A'' \to B', & {}^{**}T(f, F)(y) = F({}^{*}T(y, f)) \\ {}^{***}T : A'' \times B'' \to C'', & {}^{***}T(F, G)(f) = G({}^{**}T(f, F)) \end{array}$$

for all $x \in A, y \in B, f \in C', F \in A'', G \in B''$ (so-called the *second Arens adjoint* of T) [3].

A bilinear operator $T : A \times B \to C$ is said to be *of order bounded variation* if, for all $(x, y) \in A^{+} \times B^{+}$, the set

$$\Big\{ \sum_{n,m}^{N,M} |T(a_n, b_m)| : a_n \in A^{+}, b_m \in B^{+} \text{ and } \sum_{n=1}^{N} a_n = x, \sum_{m=1}^{M} b_m = y \ (N, M \in \mathbb{N}) \Big\}$$

is order bounded in C. T is said to be *order bounded* if for all $(x, y) \in A^{+} \times B^{+}$ we have that

$$\{T(a, b) : 0 \le a \le y, 0 \le b \le y\}$$

is order bounded. A bilinear map $T : A \times B \to C$ is said to be *separately disjointness preserving* if

$$(a_1, b_1) \perp (a_2, b_2) \text{ in } A \times B \text{ implies } T(a_1, b) \perp T(a_2, b) \text{ and } T(a, b_1) \perp T(a, b_2)$$

for all $a \in A$ and $b \in B$. A bilinear operator $T : A \times A \to A$ is called *separately band preserving* if

$$x \perp y \text{ in } A \quad \text{implies} \quad T(x, z) \perp y \text{ and } T(z, x) \perp y$$

for all $z \in A$, where $x \perp y$ means $|x| \wedge |y| = 0$. A bilinear operator $T : A \times B \to C$ is called *separately order bounded* (respectively *separately order continuous*) if the operators

$$a \mapsto T(a, y) \ (a \in A) \quad \text{and} \quad b \mapsto T(x, b) \ (b \in B)$$

are order bounded (respectively order continuous) for each $x \in A^+$ and each $y \in B^+$. If the above operators are Riesz homomorphisms, then the bilinear operator T is called a *Riesz bimorphism*. T is a Riesz bimorphism if and only if $|T(a,b)| = T(|a|,|b|)$ for all $a \in A$ and $b \in B$. We also observe that if $T : A \times B \to C$ is positive, then T is of order bounded variation, and so $T^{***} : (A')'_n \times (B')'_n \to (C')'_n$ is separately order continuous and a Riesz bimorphism (see, e.g., Theorem 2.1 in [7]).

In the sequel we make use of the fact that $T^{***}(\widehat{a},\widehat{b}) = \widehat{T(a,b)}$ for all $a \in A$ and $b \in B$. Indeed, for all $f \in A'$,

$$\begin{aligned} T^{***}(\widehat{a},\widehat{b})(f) &= \widehat{a}(T^{**}(\widehat{b},f)) = T^{**}(\widehat{b},f)(a) = \widehat{b}(T^{*}(f,a)) \\ &= T^{*}(f,a)(b) = f(T(a,b)) = \widehat{T(a,b)}(f). \end{aligned}$$

In this work we shall concentrate on the first Arens adjoint. Similar results hold for the second.

All vector lattices under consideration are Archimedean.

For the elementary theory of vector lattices and terminology not explained here we refer to [1, 13, 16].

2. The Triadjoint of an Orthosymmetric Bilinear Map

In this section we prove that the extension T^{***} of a positive orthosymmetric bilinear map $T : A \times A \to B$ is again a positive orthosymmetric bilinear map.

Definition 2.1. Let A and B be vector lattices. A bilinear map $T : A \times A \to B$ is said to be *orthosymmetric* if $x \wedge y = 0$ implies $T(x,y) = 0$ for all $x, y \in A$. T is called *positively semidefinite* if $T(x,x) \geq 0$ for all $x \in A$ and *symmetric* if $T(x,y) = T(y,x)$ for all $x, y \in A$.

Every positive orthosymmetric bilinear map $T : A \times A \to B$ is positively semidefinite; for,

$$T(x,x) = T(x^+ - x^-, x^+ - x^-) = T(x^+,x^+) + T(x^-,x^-) \geq 0,$$

as $T(x^+,x^-) = T(x^-,x^+) = 0$. Moreover, if A is Archimedean, then T is symmetric by [10, Corollary 2].

Theorem 2.2. *Let A, B be vector lattices and $T : A \times A \to B$ be positive orthosymmetric. Then the triadjoint $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ is positive orthosymmetric.*

Proof. Let T be positive orthosymmetric. Then clearly T^{***} is positive. We have to show that T^{***} is orthosymmetric. We first show that if $x \in A^+$ and $0 \leq G, H \in (A')'_n$ satisfy $G, H \leq \widehat{x}$ and $G \wedge H = 0$, then $T^{***}(G,H) = 0$, which is the main step of the proof. This result is also proved in [14] by the technique used in [4, Theorem 2.12]. Here we obtain the result by means of the approximation by components ([12]), as follows.

First we observe some notations: Let A be a vector lattice and let a be a fixed element of A. If $E := \{F \in (A')'_n : \exists\lambda > 0, |F| \leq \lambda\widehat{a}\}$-the ideal generated in $(A')'_n$ by $\widehat{a}$. Consider the Boolean algebra $\mathcal{R}$ generated by the set of all band projections of E onto principal bands generated by positive elements of $\widehat{A}$ in E. If we denote the band projection onto the band generated in E by the element $F \in E$ by P_F, then $\mathcal{R}$ is generated by the set $\mathcal{G} := \{P_{\widehat{x}} : x \in A^+\}$-the set of all band projections onto the principal ideals generated by elements $\widehat{x}$ with $x \in A^+$. Also, $\mathcal{G}\widehat{a} := \{P_{\widehat{x}}\widehat{a} : x \in A^+\}$.

Now we prove that if $0 \leq G, H \in (A')'_n$ satisfy $G, H \leq \widehat{x}$ for some $x \in A^+$ and $G \wedge H = 0$, then $T^{***}(G,H) = 0$. For this, it is sufficient to prove that $T^{***}(P_G\widehat{x}, P_H\widehat{x}) = 0$ since $0 \leq G \leq P_G\widehat{x}$ and $0 \leq H \leq P_H\widehat{x}$. (Note that, as band projections are positive, $0 \leq G \wedge H = P_GG \wedge P_HH \leq P_G\widehat{x} \wedge P_H\widehat{x}$, and so $P_G\widehat{x} \wedge P_H\widehat{x} = 0$ implies $G \wedge H = 0$. Hence $T^{***}(G,H) \leq T^{***}(P_G\widehat{x}, P_H\widehat{x})$ by the positivity of T^{***}.) But, to do this, it is sufficient to proof that $T^{***}(\widehat{x} - F, F) = 0$ for any component F of $\widehat{x}$; that is, $(\widehat{x} - F) \wedge F = 0$.

The proof of this is in four steps, as follows.

Step 1. Let $F \in \mathcal{G}\widehat{a}$, say $F = P_{\widehat{a}}\widehat{x} = \sup_n(n\widehat{a} \wedge \widehat{x})$. Then it follows from

$$\widehat{x} - F = \widehat{x} - \sup_n(n\widehat{a} \wedge \widehat{x}) = \inf_n(\widehat{x} - n\widehat{a} \wedge \widehat{x}) = \inf_n(\widehat{x} - n\widehat{a})^+$$

that for each fixed n

$$T^{***}(\widehat{x}-F,(\widehat{na}-\widehat{x})^{+}) \leq T^{***}((\widehat{x}-\widehat{na})^{+},(\widehat{na}-\widehat{x})^{+}) = T(\widehat{(x-na)^{+},(na-x)^{+}}) = 0,$$

as $(x-na)^{+} \wedge (na-x)^{+} = (x-na)^{+} \wedge (x-na)^{-} = 0$ and T is orthosymmetric. Hence

$$T^{***}(\widehat{x}-F,(\widehat{na}-\widehat{x})^{+}) = 0,$$

and so

$$nT^{***}(\widehat{x}-F,(\widehat{na}-\widehat{x})^{+}) = 0.$$

This implies that for each n

$$T^{***}(\widehat{x}-F,(\widehat{a}-\frac{1}{n}\widehat{x}))^{+} = 0.$$

Therefore

$$T^{***}(\widehat{x}-F,\widehat{a}) = 0, \quad \text{as} \quad n \to \infty.$$

It follows that for each n

$$nT^{***}(\widehat{x}-F,\widehat{a}) = 0; \quad \text{i.e., } T^{***}(\widehat{x}-F,\widehat{na}) = 0.$$

Hence,

$$0 \leq T^{***}(\widehat{x}-F,\widehat{na}\wedge\widehat{x}) \leq T^{***}(\widehat{x}-F,\widehat{na}) = 0;$$

$$\text{i.e., } T^{***}(\widehat{x}-F,\widehat{na}\wedge\widehat{x}) = 0.$$

Since this holds for each n, we get

$$\sup_n T^{***}(\widehat{x}-F,\widehat{na}\wedge\widehat{x}) = 0,$$

which leads that, by the separate order continuity of T^{***} as T is positive,

$$T^{***}(\widehat{x}-F,F) = T^{***}(\widehat{x}-F,\sup_n(\widehat{na}\wedge\widehat{x})) = \sup_n T^{***}(\widehat{x}-F,\widehat{na}\wedge\widehat{x}) = 0;$$

that is,

$$T^{***}(\widehat{x}-F,F) = 0.$$

Step 2. Let $F = \bigwedge_{i=1}^{m} F_i$ where either $F_i \in \mathcal{G}\widehat{a}$ or $\widehat{x}-F_i \in \mathcal{G}\widehat{a}$. Then

$$\widehat{x}-F = \bigvee_{i=1}^{m}(\widehat{x}-F_i),$$

and so

$$\begin{aligned} 0 \leq T^{***}(\widehat{x}-F,F) &= T^{***}(\bigvee_{i=1}^{m}(\widehat{x}-F_i),\bigwedge_{i=1}^{m}F_i) \\ &\leq T^{***}(\sum_{i=1}^{m}(\widehat{x}-F_i),F_i) \\ &= \sum_{i=1}^{m}T^{***}((\widehat{x}-F_i),F_i) \\ &= 0 \qquad \text{(by Step 1)}; \end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x}-F,F) = 0.$$

Step 3. Let $F = \bigvee_{i=1}^{n} F_i$ where each F_i is of the form F had in Step 1 (that is, $F_i = \bigwedge_{j=1}^{m} F_{ij}, \forall i = 1, 2, \cdots, n$, and so $F = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} F_{ij}$). Then, in the same way as Step 2,

$$\widehat{x} - F = \bigwedge_{i=1}^{m} (\widehat{x} - F_i),$$

and so

$$\begin{aligned}
0 \leq T^{***}(\widehat{x} - F, F) &= T^{***}(\bigwedge_{i=1}^{m} (\widehat{x} - F_i), \bigvee_{i=1}^{m} F_i) \\
&\leq T^{***}(\widehat{x} - F_i, \bigvee_{i=1}^{m} F_i) \\
&\leq T^{***}(\widehat{x} - F_i, \sum_{i=1}^{m} F_i) \\
&= \sum_{i=1}^{m} T^{***}(\widehat{x} - F_i, F_i) \\
&= 0 \qquad \text{(by Step 2);}
\end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x} - F, F) = 0.$$

Step 4. Let $F \in \mathcal{R}\widehat{x}$, If $F = \sup_\alpha F_\alpha$ or $F = \inf_\alpha F_\alpha$ with each F_α is a component of $\widehat{x}$ (that is, $(\widehat{x} - F_\alpha) \wedge F_\alpha = 0$ for each α) having the property that $T^{***}(\widehat{x} - F_\alpha, F_\alpha) = 0$, then using the separate order continuity of T^{***} we show that F has the same property;

$$\text{i.e., } T^{***}(\widehat{x} - F, F) = 0.$$

Indeed, suppose that $F = \sup_\alpha F_\alpha$. For each fixed β

$$\begin{aligned}
T^{***}(\widehat{x} - F\alpha, F_\beta) &= T^{***}(\widehat{x} - F_\beta, F_\beta) + T^{***}(F_\beta - F_\alpha, F_\beta) \\
&= 0 + T^{***}(F_\beta - F_\alpha, F_\beta) \qquad \text{(by hypothesis)} \\
&= T^{***}(F_\beta, F_\beta) - T^{***}(F_\alpha, F_\beta);
\end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x} - F_\alpha, F_\beta) = T^{***}(F_\beta, F_\beta) - T^{***}(F_\alpha, F_\beta) \qquad \forall \alpha. \tag{1}$$

Since $F = \sup_\alpha F_\alpha$, $-F = \inf_\alpha(-F_\alpha)$, and so, by the separate order continuity of T^{***},

$$\begin{aligned}
\inf_\alpha(-T^{***}(F_\alpha, F_\beta)) &= \inf_\alpha T^{***}(-F_\alpha, F_\beta) \\
&= T^{***}(\inf_\alpha(-F_\alpha), F_\beta) \\
&= T^{***}(-F, F_\beta) \qquad \text{(fixed } \beta\text{);}
\end{aligned}$$

$$\text{i.e., } \inf_\alpha(-T^{***}(F_\alpha, F_\beta)) = T^{***}(-F, F_\beta).$$

Hence

$$\inf_\alpha(T^{***}(F_\beta, F_\beta) - T^{***}(F_\alpha, F_\beta)) = T^{***}(F_\beta, F_\beta) - T^{***}(F, F_\beta),$$

and so it follows from (1) that

$$\inf_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta) = T^{***}(F_\beta - F, F_\beta) \qquad \text{(fixed } \beta\text{).} \tag{2}$$

This holds for all β, and moreover we have

$$\sup_\beta T^{***}(F_\beta - F, F_\beta) = 0.$$

To see this, first note that $F = \sup_\beta F_\beta$, and so $F_\beta - F \leq 0$ for each β. Hence

$$T^{***}(F_\beta - F, F_\beta) \leq 0.$$

since T^{***} is positive. Now take some fixed β_0. Then $\forall \beta \geq \beta_0$ we have $F_{\beta_0} \leq F_\beta$ (as $F_\beta \uparrow$). Therefore, again since T^{***} is positive,

$$T^{***}(F_\beta - F, F_{\beta_0}) \leq T^{***}(F_\beta - F, F_\beta) \leq 0.$$

Using the separate order continuity of T^{***}, we have

$$\begin{aligned} 0 = T^{***}(F - F, F_{\beta_0}) &= T^{***}(\sup_\beta F_\beta - F, F_{\beta_0}) = \sup_\beta T^{***}(F_\beta - F, F_{\beta_0}) \\ &\leq \sup_\beta T^{***}(F_\beta - F, F_\beta) \leq 0; \end{aligned}$$

$$\text{i.e., } \sup_\beta T^{***}(F_\beta - F, F_\beta) = 0.$$

It follows from this and (2) that

$$\sup_\beta(\inf_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) = 0. \tag{3}$$

Also, again using the separate order continuity of T^{***}, for fixed β

$$\begin{aligned} \inf_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) &= T^{***}(\inf_\alpha(\widehat{x} - F_\alpha), F_\beta) \\ &= T^{***}((\widehat{x} - \sup_\alpha F_\alpha), F_\beta) \\ &= T^{***}((\widehat{x} - F), F_\beta); \end{aligned}$$

$$\text{i.e., } \inf_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta) = T^{***}(\widehat{x} - F, F_\beta).$$

This is true for all β, and also for all α. Therefore, by (3) and once more using the separate order continuity of T^{***}, we get

$$\begin{aligned} 0 = \sup_\beta(\inf_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) &= \sup_\beta T^{***}(\widehat{x} - F, F_\beta) \\ &= T^{***}(\widehat{x} - F, \sup_\beta F_\beta) \\ &= T^{***}(\widehat{x} - F, F) \end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x} - F, F) = 0.$$

In exactly the same way above we now show that if $F = \inf_\alpha F_\alpha$ such that $(\widehat{x} - F_\alpha) \wedge F_\alpha = 0$ and $T^{***}(\widehat{x} - F_\alpha, F_\alpha) = 0$ for each α, then

$$\text{i.e., } T^{***}(\widehat{x} - F, F) = 0.$$

Let β be fixed. Then

$$\begin{aligned} T^{***}(\widehat{x} - F_\alpha, F_\beta) &= T^{***}(\widehat{x} - F_\beta, F_\beta) + T^{***}(F_\beta - F_\alpha, F_\beta) \\ &= 0 + T^{***}(F_\beta - F_\alpha, F_\beta) \quad \text{(by hypothesis)} \\ &= T^{***}(F_\beta, F_\beta) - T^{***}(F_\alpha, F_\beta); \end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x} - F_\alpha, F_\beta) = T^{***}(F_\beta, F_\beta) - T^{***}(F_\alpha, F_\beta) \qquad \forall \alpha. \tag{4}$$

Since $F = \inf_\alpha F_\alpha$, $-F = \sup_\alpha(-F_\alpha)$, and so, by the separate order continuity of T^{***},

$$\begin{aligned}\sup_\alpha(-T^{***}(F_\alpha, F_\beta)) &= \sup_\alpha T^{***}(-F_\alpha, F_\beta)\\ &= T^{***}(\sup_\alpha(-F_\alpha), F_\beta)\\ &= T^{***}(-F, F_\beta) \quad \text{(fixed } \beta);\end{aligned}$$

$$\text{i.e., } \sup_\alpha(-T^{***}(F_\alpha, F_\beta)) = T^{***}(-F, F_\beta).$$

Hence

$$\sup_\alpha(T^{***}(F_\beta, F_\beta) - T^{***}(F_\alpha, F_\beta)) = T^{***}(F_\beta, F_\beta) - T^{***}(F, F_\beta),$$

and so it follows from (4) that

$$\sup_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta) = T^{***}(F_\beta - F, F_\beta) \quad \text{(fixed } \beta). \tag{5}$$

This holds for all β, and moreover we have

$$\inf_\beta T^{***}(F_\beta - F, F_\beta) = 0.$$

To see this, first note that $F = \inf_\beta F_\beta$, and so $F_\beta - F \geq 0$ for each β. Hence

$$T^{***}(F_\beta - F, F_\beta) \geq 0.$$

since T^{***} is positive. Now take some fixed β_0. Then $\forall \beta \geq \beta_0$ we have $F_\beta \leq F_{\beta_0}$ (as $F_\beta \downarrow$). Therefore, again since T^{***} is positive,

$$0 \leq T^{***}(F_\beta - F, F_\beta) \leq T^{***}(F_\beta - F, F_{\beta_0}).$$

Using the separate order continuity of T^{***}, we have

$$\begin{aligned}0 = T^{***}(\inf_\beta F_\beta - F, F_\beta) &= \inf_\beta T^{***}(F_\beta - F, F_\beta) \leq \inf_\beta T^{***}(F_\beta - F, F_{\beta_0})\\ &= T^{***}(\inf_\beta F_\beta - F, F_{\beta_0}) = 0;\end{aligned}$$

$$\text{i.e., } \inf_\beta T^{***}(F_\beta - F, F_\beta) = 0.$$

It follows from this and (5) that

$$\inf_\beta(\sup_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) = 0. \tag{6}$$

Also, again using the separate order continuity of T^{***}, for fixed β

$$\begin{aligned}\sup_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) &= T^{***}(\sup_\alpha(\widehat{x} - F_\alpha), F_\beta)\\ &= T^{***}((\widehat{x} - \inf_\alpha F_\alpha), F_\beta)\\ &= T^{***}((\widehat{x} - F), F_\beta);\end{aligned}$$

$$\text{i.e., } \sup_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) = T^{***}(\widehat{x} - F, F_\beta).$$

This is true for all β, and also for all α. Therefore, by (6) and once more using the separate order continuity of T^{***}, we get

$$\begin{aligned}0 = \inf_\beta(\sup_\alpha T^{***}(\widehat{x} - F_\alpha, F_\beta)) &= \inf_\beta T^{***}(\widehat{x} - F, F_\beta)\\ &= T^{***}(\widehat{x} - F, \inf_\beta F_\beta)\\ &= T^{***}(\widehat{x} - F, F)\end{aligned}$$

$$\text{i.e., } T^{***}(\widehat{x} - F, F) = 0,$$

from which the result follows. □

Summarizing we have proved so far that if $0 \leq G, H \in (A')'_n$ satisfy $G, H \leq \widehat{x}$ for some $x \in A^+$ and $G \wedge H = 0$, then $T^{***}(G, H) = 0$. In the general case when $0 \leq G, H \in (A')'_n$ be arbitrary such that $G \wedge H = 0$, the result follows from the fact that the band $I_{\widehat{A}} = \{F \in (A')'_n : |F| \leq \widehat{x} \text{ for some } x \in A^+\}$ generated by $\widehat{A}$ is order dense in $(A')'_n$ (for details see [14, Theorem 1]).

Our aim now is to extend the above result; in other words, we aim to prove that if A, B are vector lattices and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then the bilinear map $T^{***} : A'' \times A'' \to B''$ is positive orthosymmetric. First we require some preliminaries.

Lemma 2.3. *Let A, B be vector lattices and $0 \leq f \in B'$. If $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then*

(i) $(f(T(x, y)))^2 \leq f(T(x, x)) \cdot f(T(y, y)), \quad \forall x, y \in A.$

(ii) $(T^{***}(F, G)(f))^2 \leq T^{***}(F, F)(f) \cdot T^{***}(G, G)(f), \quad \forall F, G \in (A')'_n.$

Proof. **(i)** Given $0 \leq f \in B'$, it is easy to see that the map

$$(x, y) \mapsto f(T(x, y))$$

is a positive orthomorphism bilinear form (that is, the map $f \circ T : A \times A \to \mathbb{R}$ is a positive orthomorphism bilinear map) and so it is positively semidefinite and symmetric. Now the result follows from the Cauchy-Schwarz Inequality (see, e.g., [10]).

(ii) Since $T^{***} : (A')'_n \times (A')'_n \to (A')'_n$ a positive orthomorphism bilinear map by Theorem 2.2, in (i) replacing A by $(A')'_n$ we see that, given $0 \leq f \in A'$, the map

$$(F, G) \mapsto \widehat{f}(T^{***}(F, G))$$

is a positive orthosymmetric bilinear form (that is, the map $\widehat{f} \circ T^{***} : (A')'_n \times (A')'_n \to \mathbb{R}$ is a positive orthomorphism bilinear map), where $\widehat{f}(F) := F(f)$ for all $F \in (A')'_n$ as usual, and satisfies the Cauchy-Schwarz Inequality. □

Lemma 2.4. *Let A, B be vector lattices and $0 \leq f \in B'$. If $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then for all $x \in A$*

(i) $(T^*(f, x))^+ = T^*(f, x^+)$.

(ii) *The map $T_f : A \to A'$ defined by $T_f(x) = T^*(f, x)$ is a Riesz homomorphism and the adjoint $T'_f : A'' \to A'$ of T_f satisfies $T'_f(F) = T^{**}(F, f)$ for all $F \in A''$.*

Proof. **(i)** Since T is positive, $T(x^+, y) \geq T(x, y)$ for all $y \in A$. Hence

$$T^*(f, x^+)(y) = f(T(x^+, y)) \geq f(T(x, y)) = T^*(f, x)(y)$$

for all $y \in A$, and so $T^*(f, x^+) \geq T^*(f, x)$. Also, since T^* is positive by [7, Theorem 2.1], we have $T^*(f, x^+) \geq 0$. It follows that

$$T^*(f, x^+) \geq T^*(f, x) \vee 0 = (T^*(f, x))^+.$$

Conversely, let $y \in A^+$. It follows from

$$x^+ \wedge (x^- \wedge y) = 0 \quad \text{and} \quad (x^- - x^- \wedge y) \wedge (y - x^- \wedge y) = 0$$

that

$$T(x^+, x^- \wedge y) = 0 \quad \text{and} \quad T(x^- - x^- \wedge y, y - x^- \wedge y) = 0$$

since T is orthosymmetric. Hence

$$T(x^-, y - x^- \wedge y) - T(x^- \wedge y, y - x^- \wedge y) = 0,$$

and so, by the positivity of T,

$$T(x^-, y - x^- \wedge y) = T(x^- \wedge y, y - x^- \wedge y) \leq T(y, y);$$

$$\text{i.e., } T(x^-, y - x^- \wedge y) \leq T(y, y).$$

Therefore, since $0 \leq f \in B'$,

$$\begin{aligned}
(T^*(f,x))^+(y) &\geq (T^*(f,x))^+(y - x^- \wedge y) \geq T^*(f,x)(y - x^- \wedge y) \\
&= f(T(x, y - x^- \wedge y)) = f(T(x^+ - x^-, y - x^- \wedge y) \\
&= f(T(x^+, y) - T(x^+, x^- \wedge y) - T(x^-, y) + T(x^-, x^- \wedge y)) \\
&= f(T(x^+, y) - (T(x^-, y) - T(x^-, x^- \wedge y))) \\
&= f(T(x^+, y) - (T(x^-, y) + T(x^-, -x^- \wedge y))) \\
&= f(T(x^+, y) - T(x^-, y - x^- \wedge y)) \\
&= f(T(x^+, y)) - f(T(x^-, y - x^- \wedge y)) \\
&\geq f(T(x^+, y)) - f(T(y, y)) \\
&= T^*(f, x^+))(y) - (T^*(f, y))(y).
\end{aligned}$$

Now replacing $\frac{1}{n}y$ by y we obtain

$$(T^*(f,x))^+(y) \geq (T^*(f,x^+))(y) - \frac{1}{n}(T^*(f,y))(y) \qquad (n = 1, 2, \cdots),$$

and so

$$(T^*(f,x))^+(y) \geq (T^*(f,x^+))(y), \quad \text{as} \quad n \to \infty.$$

Since this hold for all $y \in A^+$, we have

$$(T^*(f,x))^+ \geq T^*(f,x^+), \quad \text{as} \quad n \to \infty.$$

This proves that $(T^*(f,x))^+ = T^*(f,x^+)$, as required.

(ii) That the map T_f is a Riesz homomorphism follows immediately from (i). Note that

$$(T_f'(F))(x) = F(T_f(x)) = F(T^*(f,x)) = T^{**}(F,f)(x)$$

for all $x \in A$ and $F \in A''$; that is, $T_f'(F) = T^{**}(F,f)$ for all $F \in A''$. □

Lemma 2.5. *Let A, B be vector lattices, $T : A \times A \to B$ be a positive orthosymmetric bilinear map, $0 \leq f \in B'$ and $0 \leq F \in A''$. If $g \in A'$ satisfies $0 \leq g \in T^{**}(F,f)$, then $g = T^{**}(G,f)$ for some $0 \leq G \leq F \in A''$. Also, if $F \in (A')_n'$, then $G \leq\in (A')_n'$. That is, the adjoint T_f', as defined above, is interval preserving.*

Proof. Since the adjoint T_f' of the Riesz homomorphism T_f is interval preserving (see, e.g., [1, Theorems 7.4 and 7.8]), we have

$$T_f'[0,F] = [0, T_f'(F)] = [0, T^{**}(F,f)].$$

Therefore, if $0 \leq g \in T^{**}(F,f)$, there exists some $0 \leq G \leq F \in A''$ such that $g = T_f'(G) = T^{**}(G,f)$. We observe that if $0 \leq G \leq F \in A''$ with $F \in (A')_n'$, then $G \in (A')_n'$ since $(A')_n'$ is solid (indeed a band in A''). □

Lemma 2.6. *Let A, B be vector lattices and $T : A \times A \to B$ be a positive orthosymmetric bilinear map. If $F_\alpha \downarrow 0$ in A'', then*

(i) *$T^{***}(F_\alpha, F_\alpha) \downarrow 0$.*

(ii) *$T^{***}(F, F_\alpha) \downarrow 0$ for all $0 \leq F \in A''$.*

Proof. **(i)** Let $0 \leq f \in B\prime$ and β be fixed. Then for all $\alpha \geq \beta$ we have $F_\alpha \leq F_\beta$ as $F_\alpha \downarrow$, and so $F_\alpha(f) \downarrow$. In particular, for $0 \leq T^{**}(F_\beta, f) \in A'$,

$$F_\alpha(T^{**}(F_\beta, f)) \downarrow 0 \quad \text{ i.e., } T^{***}(F_\alpha, F_\beta)(f) \downarrow 0$$

for all $0 \leq f \in B\prime$. Therefore we have $T^{***}(F_\alpha, F_\beta) \downarrow 0$. it follows from

$$0 \leq T^{***}(F_\alpha, F_\alpha) \leq T^{***}(F_\alpha, F_\beta)$$

that

$$0 \leq \inf_\alpha T^{***}(F_\alpha, F_\alpha) \leq \inf_\alpha T^{***}(F_\alpha, F_\beta) = 0; \quad \text{i.e., } T^{***}(F_\alpha, F_\alpha) \downarrow 0.$$

(ii) Let $0 \leq f \in B\prime$ and $F_\alpha \downarrow 0 \in (A')'_n$. Then it follows from (i) that

$$T^{***}(F_n, F_n)(f) \leq \frac{1}{n^4} \quad (n = 1, 2, \cdots)$$

for some subsequence (F_n) of the net (F_α). By Lemma 2.3, for all $x \in A^+$ we have

$$0 \leq T^{***}(F_n, f)(x) \leq \sqrt{T^{***}(F_n, F_n)(f)} \cdot \sqrt{f(T(x, x))} \quad (x \in A^+),$$

and so

$$\begin{aligned} 0 \leq \sum_{n=1}^{\infty} T^{**}(F_n, f)(x) &\leq \sum_{n=1}^{\infty} \Big(\sqrt{T^{***}(F_n, F_n)(f)} \cdot \sqrt{f(T(x, x))} \Big) \\ &\leq \sqrt{f(T(x, x))} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &< \infty; \end{aligned}$$

that is, the series $\sum_{n=1}^{\infty} T^{**}(F_n, f)(x)$ is convergent in $\mathbb{R}^+$. So the functional $g : A^+ \to \mathbb{R}^+$ defined by

$$g(x) = \sum_{n=1}^{\infty} T^{**}(F_n, f)(x) \quad (x \in A^+),$$

is additive on A^+ and has a positive linear extension (see, e.g., [1, Theorem 1.7]) to the whole of A, which is denoted by g again.

We have to show that $T^{***}(F, F_n)(f) \downarrow 0$. So let $m \in \mathbb{N}$ be arbitrary. Then

$$\begin{aligned} \sum_{n=1}^{m} T^{***}(F, F_n)(f) &= \sum_{n=1}^{m} F(T^{**}(F_n, f)) \\ &= F\Big(\sum_{n=1}^{m} (T^{**}(F_n, f)) \Big) \\ &\leq F\Big(\sum_{n=1}^{\infty} T^{**}(F_n, f) \Big) \\ &= F(g), \end{aligned}$$

which implies

$$\sum_{n=1}^{\infty} T^{***}(F, F_n)(f) \leq F(g), \quad \text{ as } m \to \infty.$$

This shows that the series $\sum_{n=1}^{\infty} T^{***}(F, F_n)(f)$ is convergent in $\mathbb{R}^+$, and so

$$\lim_{n \to \infty} T^{***}(F, F_n)(f) = 0.$$

It follows that $T^{***}(F, F_n)(f) \downarrow 0$ for all $0 \leq f \in B\prime$, and so $T^{***}(F, F_n) \downarrow 0$. $\square$

Theorem 2.7. *If A, B are vector lattices and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then $T^{***}(F, G) = 0$ for all $F \in (A')'_s$ and $G \in (A')'_n$.*

Proof. Suppose that $0 \le F \in (A')'_s$ and $0 \le G \in (A')'_n$. Fix $0 \le f \in B\prime$ and set

$$J = \{L \in (A')'_n : L \ge 0,\ T^{***}(F, L)(f) = F(T^{**}(L, f)) = 0\}.$$

Then J is a solid subspace of the positive cone of $(A')'_n$. It is easily seen that J is closed under addition and multiplication by positive scalars. Now if $0 \le L_2 \le L_1$ in $(A')'_n$ with $L_1 \in J$, then we have $0 \le T^{***}(F, L_2) \le T^{***}(F, L_1)$. Hence

$$0 \le T^{***}(F, L_2)(f) \le T^{***}(F, L_1)(f) = 0$$

for all $0 \le f \in A'$, which shows that $L_2 \in J$. Also if $L_1, L_2 \in J$, then it follows from $0 \le L_1 \vee L_2 \le L_1 + L_2$ that

$$0 \le T^{***}(F, L_1 \vee L_2) \le T^{***}(F, L_1 + L_2) = T^{***}(F, L_1) + T^{***}(F, L_2).$$

Hence for all $0 \le f \in A'$ we have

$$0 \le T^{***}(F, L_1 \vee L_2)(f) \le T^{***}(F, L_1)(f) + T^{***}(F, L_2)(f) = 0 + 0 = 0;$$

that is, $L_1 \vee L_2$. This shows that J is a lattice ideal in the positive cone of $(A')'_n$ Moreover, J is order dense in the positive cone of $(A')'_n$; that is, for each $0 < M \in (A')'_n$ there exist an element $L \in J$ such that $0 < L \le M$), as follows.

If $0 < M \in (A')'_n$ and $T^{***}(F, M)(f) = 0$, then we may choose $L = M$. Therefore assume that $T^{***}(F, M)(f) > 0$, i.e., $F(T^{**}(M, f) > 0$ (note that $F \neq 0$). Since $F \in (A')'_s$, $N_F^{dd} = A'$ ($\Leftrightarrow$ $N_F^d = A'$), i.e., N_F is order dense in A' since A' is Archimedean by [4, Theorem 1.1 (iii)]. Hence there exists $g \in N_F$ (i.e., $H(g) = 0$) such that $0 < g \le T^{**}(M, f)$. Note that F is singular; that is, $F \in (A')'_s = ((A')'_n)^d$ and $F \neq 0$. Thus, by Lemma 2.5, there exists $L \in (A')'_n$ with $0 < L \le M$ such that $g = T^{**}(L, f)$. This implies that $L \in J$ since $F(T^{**}(L, f)) = F(g) = 0$. Note that clearly it has to be that $L > 0$ as $g > 0$.

Now we consider the set

$$J_G = \{L \in J : L \le G\}.$$

It follows that J_G is an upwards directed net (i.e., $J_G \uparrow$) in J, which is bounded above by G. Since $(A')'_n$ is Dedekind complete, there exists an element G_0 in $(A')'_n$ such that $G_0 = \sup J_G$ and $G_0 \le G$. Observe that $0 \le G_0 - L$ for all $L \in J_G$ since $J_G \uparrow$, and so $(G_0 - L) \downarrow_{J_G} 0$ in $(A')'_n$. It follows from Lemma 2.6 (ii) that

$$0 \le T^{***}(F, G_0 - L) \downarrow_{J_G} 0$$

for all $0 \le F \in (A')'_s$, and so

$$0 \le T^{***}(F, G_0 - L)(f) \downarrow_{J_G} 0$$

for all $0 \le f \in B\prime$. This implies immediately that $T^{***}(F, G_0 - L)(f) = 0$ since $L \in J$. Hence $G_0 \in J$, which shows that $G_0 \in J_G$ since $G_0 \le G$. Now we claim that $G_0 = G$. Suppose that $G - G_0 > 0$. Since J is order dense in the positive cone of $(A')'_n$, there exists $L \in J$ such that $0 < L \le G - G_0$. Thus $0 < L + G_0 \le G$, and so $L + G_0 \in J_G$ since $L + G_0 \in J$. But $G_0 = \sup J_G$, and so we have $L + G_0 \le G_0$. This is a contradiction since $L > 0$. Therefore $G = G_0 \in J$. Hence $T^{***}(F, G)(f) = 0$ for all $0 \le f \in B\prime$, showing that $T^{***}(F, G) = 0$ for all $0 \le F \in (A')'_s$ and $0 \le G \in (A')'_n$.

The general case follows from the decompositions F and G into positive and negative parts; for, if $F \in (A')'_s$ and $G \in (A')'_n$, then

$$\begin{aligned} T^{***}(F, G) &= T^{***}(F^+ - F^-, G^+ - G^-) \\ &= T^{***}(F^+, G^+) - T^{***}(F^+, G^-) - T^{***}(F^-, G^+) + T^{***}(F^-, G^-) \\ &= 0, \end{aligned}$$

as required. □

Corollary 2.8. *If A, B are vector lattices and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then $T^{***}(F, G) = 0$ for all $F \in A''$ and $G \in (A')'_s$. In particular, $T^{***}(G, G) = 0$ for all $G \in (A')'_s$. In other words,*

$$T^{***}|_{A'' \times (A')'_s} = 0 \quad (\textit{hence } T^{***}|_{(A')'_s \times (A')'_s} = 0 \textit{ and } T^{***}|_{(A')'_n \times (A')'_s} = 0).$$

Proof. Let $G \in (A')'_s$. It follows from the fact that $\widehat{x} \in (A')'_n$ for all $x \in A$ that $T^{**}(G, \widehat{x}) = 0$ by Theorem 2.7. Since T is a symmetric bilinear map, we have

$$(T^{**}(G, f))(x) = \widehat{x}(T^{**}(G, f)) = T^{**}(\widehat{x}, G)(f) = T^{***}(G, \widehat{x})(f) = 0$$

for all $x \in A$ and $f \in B'$. Hence $T^{**}(G, f) = 0$ for all $f \in B'$, and so

$$T^{***}(F, G)(f) = F(T^{**}(G, f) = F(0) = 0$$

for all $f \in A'$ and $F \in A''$. Therefore $T^{***}(F, G) = 0$ for all $F \in A''$, and so the result holds for all $F \in A''$ and $G \in (A')'_s$. □

Before moving on to the main result of this section, we remark that $(A')'_n$ and its disjoint complement $(A')'_s$ are band in the Dedekind complete vector lattice A'', and so they themselves are Dedekind complete vector lattices. Hence the *order direct sum* is

$$A'' = (A')'_n \oplus (A')'_s.$$

Theorem 2.9. *If A, B are vector lattices and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then the bilinear map $T^{***} : A'' \times A'' \to B''$ is positive orthosymmetric.*

Proof. Clearly if the bilinear map T is positive, then so is T^{***}.

Let $0 \leq F, G \in A''$ and $F \wedge G = 0$. By the order direct sum of A'', we decompose $F, G \in A''$ as

$$F = F_n + F_s \quad \text{and} \quad G = G_n + G_s,$$

where $0 \leq F_n, G_n \in (A')'_n$ and $0 \leq F_s, G_s \in (A')'_s$. Then it follows from from Theorem 2.7 and Corollary 2.8 that

$$\begin{aligned} T^{***}(F, G) &= T^{***}(F_n, G_n) + T^{***}(F_n, G_s) + T^{***}(F_s, G_n) + T^{***}(F_s, G_s) \\ &= T^{***}(F_n, G_n) + 0 + 0 + 0 \\ &= T^{***}(F_n, G_n); \end{aligned}$$

$$\text{i.e., } T^{***}(F, G) = T^{***}(F_n, G_n).$$

But $F_n \wedge G_n = 0$ since $0 \leq F_n \wedge G_n \leq F_n + F_s \wedge G_n + G_s = F \wedge G = 0$, and so $T^{***}(F_n, G_n) = 0$ since the bilinear map $T^{***} : (A')'_n \times (A')'_n \to (B')'_n$ is is orthosymmetric by Theorem 2.2. Therefore we have $T^{***}(F, G) = 0$, as required. □

Corollary 2.10. *If A, B are vector lattices and $T : A \times A \to B$ is a positive orthosymmetric bilinear map, then $T^{***}(F, G) \in (A')'_n$ for all $F, G \in A''$; in other words, $T^{***}(A'' \times A'') \subset (B')'_n$.*

Proof. The result is true for the positive elements of A'' as in the proof of the previous theorem. For the general case, let $F, G \in A''$ be arbitrary. Then $F = F^+ - F^-$ and $G = G^+ - G^-$ with $0 \leq F^+, F^-, G^+, G^- \in A'' = (A')'_n \oplus (A')'_s$, and so

$$\begin{array}{rl} F^+ = F^+_n + F^+_s & \\ F^- = F^-_n + F^-_s & 0 \leq F^+_n, F^-_n, G^+_n, G^-_n \in (A')'_n \\ G^+ = G^+_n + G^+_s & 0 \leq F^+_s, F^-_s, G^+_s, G^-_s \in (A')'_s \\ G^- = G^-_n + G^-_s . & \end{array}$$

Hence

$$\begin{array}{c} F = (F^+_n + F^+_s) - (F^-_n + F^-_s) = (F^+_n - F^-_n) + (F^+_s - F^-_s) \\ G = (G^+_n + G^+_s) - (G^-_n + G^-_s) = (G^+_n - G^-_n) + (G^+_s - G^-_s). \end{array}$$

We also have that

$$\begin{aligned} F_n \wedge F_s &= 0 \\ G_n \wedge G_s &= 0, \end{aligned}$$

where

$$\begin{array}{lll} F_n := F_n^+ - F_n^-, \quad G_n := G_n^+ - G_n^- & \in & (A')'_n \\ F_s := F_s^+ - F_s^-, \quad G_s := G_s^+ - G_s^- & \in & (A')'_s. \end{array}$$

As in the proof of preceding theorem, it follows that

$$T^{***}(F, G) = T^{***}(F_n + F_s, G_n + G_s) = T^{***}(F_n, G_n).$$

□

3. The Triadjoint of a Bi-Orthomorphism

In this section we prove that if A is a vector lattice and $T : A \times A \to A$ is a bi-orthomorphism, then so is

1. the bilinear map $T^{***} : (A')'_n \times (A')'_n \to (A')'_n$ by the technique used in [12].
2. the bilinear map $T^{***} : A'' \times A'' \to A''$ by the technique used in [4], which generalizes the first.

We observe that every separately band preserving bilinear operator is disjointness preserving. A separately band preserving bilinear operator which is also separately order bounded is called a *bi-orthomorphism* and the set of all bi-orthomorphisms of $A \times A$ into A is denoted by $Orth(A, A)$. We also observe that every bi-orthomorphism is a Riesz bimorphism, and so is positive. Hence every bi-orthomorphism is of order bounded variation, and so order bounded [11]. It follows that if $T : A \times A \to A$ is a bi-orthomorphism, then $T^{***} : (A')'_n \times (A')'_n \to (A')'_n$ is separately order continuous and a Riesz bimorphism since T is positive. So we can give the following

Definition 3.1. Let A be a vector lattice. A separately order bounded bilinear map $T : A \times A \to A$ is said to be a *bi-orthomorphism* if $x \wedge y = 0$ in A implies $T(z, x) \wedge y = 0$ for all $z \in A^+$.

Lemma 3.2. *Let A be a vector lattice and $T : A \times A \to A$ be a bilinear map such that $x \perp y$ in A implies $T(z, x) \perp y$ for all $z \in A$. Then T is orthosymmetric. In particular, every bi-orthomorphism is orthosymmetric (and hence symmetric).*

Proof. Suppose $x \perp y$ in A. Then $T(x, y) \perp x$ and $T(x, y) \perp y$, and so $T(x, y) \in \{x\}^{dd} \cap \{y\}^{dd} = \{0\}$ since $x \perp y$. Therefore $T(x, y) = 0$ whenever $x \perp y$, as required. □

Theorem 3.3. *Let A be vector lattices and $T : A \times A \to A$ be a bi-orthomorphism. Then the bilinear map $T^{***} : (A')'_n \times (A')'_n \to (A')'_n$ is a bi-orthomorphism.*

Proof. Let T be a bi-orthomorphism. Then we first show that if $x \in A^+$ and $0 \le G, H \in (A')'_n$ satisfy $F, G, H \le \widehat{x}$ and $G \wedge H = 0$, then $T^{***}(F, G) \wedge H = 0$, which is the main step of the proof. To do this, it is sufficient to proof that for any component F of $\widehat{x}$; that is, $\widehat{x} - F \wedge F = 0$, we have

$$\widehat{x} - F \wedge T^{***}(\widehat{x} - F) = 0.$$

The proof of this is in four steps, as follows.

Step 1. Suppose that $F \in \mathcal{G}\widehat{a}$, where $\mathcal{G}$ is as before, say $F = P_{\widehat{a}}\widehat{x} = \sup_n (n\widehat{a} \wedge \widehat{x})$. Then it follows from

$$\widehat{x} - F = \widehat{x} - \sup_n (n\widehat{a} \wedge \widehat{x}) = \inf_n (\widehat{x} - n\widehat{a} \wedge \widehat{x}) = \inf_n (\widehat{x} - n\widehat{a})^+$$

that for each fixed n

$$0 \le (\widehat{x} - F) \wedge (n\widehat{a} - \widehat{x})^+) \le (\widehat{x} - n\widehat{a})^+ \wedge (n\widehat{a} - \widehat{x})^+) = 0;$$

$$\text{i.e., } (\widehat{x}-F) \wedge (\widehat{na}-\widehat{x})^{+}) = 0.$$

$$\begin{aligned}
0 \le (\widehat{x}-F) \wedge T^{***}((\widehat{x},(\widehat{na}-\widehat{x})^{+}) &\le (x-na)^{+} \wedge T^{***}((\widehat{x},(\widehat{na}-\widehat{x})^{+})\\
&= (\widehat{x-na})^{+} \wedge T^{***}(\widehat{x},(\widehat{na-x})^{+})\\
&= (\widehat{x-na})^{+} \wedge T^{***}(x,\widehat{(na-x)}^{+})\\
&= (x-na)^{+} \wedge \widehat{T^{***}}(x,(na-x)^{+})\\
&= 0
\end{aligned}$$

since $(x-na)^{+} \wedge (na-x)^{+} = (x-na)^{+} \wedge (x-na)^{-} = 0$ and T is a bi-orthomorphism. Hence for each fixed n

$$(\widehat{x}-F) \wedge T^{***}((\widehat{x},(\widehat{na}-\widehat{x})^{+}) = 0,$$

and so

$$n(\widehat{x}-F) \wedge T^{***}((\widehat{x},(\widehat{na}-\widehat{x})^{+}) = 0.$$

This implies that for each n

$$(\widehat{x}-F) \wedge T^{***}(\widehat{x},(\widehat{a}-\frac{1}{n}\widehat{x})^{+}) = 0.$$

Therefore

$$(\widehat{x}-F) \wedge T^{***}(\widehat{x},\widehat{a}) = 0, \quad \text{as} \quad n \to \infty.$$

It follows that for each n

$$(\widehat{x}-F) \wedge nT^{***}(\widehat{x},\widehat{a}) = 0.$$

Hence it follows from this and the fact that T^{***} a Riesz bimorphism as observed earlier that

$$\begin{aligned}
0 \le (\widehat{x}-F) \wedge T^{***}((\widehat{x},\widehat{na}-\widehat{x}) &= (x-na)^{+} \wedge (nT^{***}((\widehat{x},\widehat{a}) \wedge T^{***}(\widehat{x},\widehat{x})\\
&= ((\widehat{x-na})^{+} \wedge nT^{***}(\widehat{x},\widehat{a})) \wedge T^{***}(\widehat{x},\widehat{x})\\
&= 0 + T^{***}(\widehat{x},\widehat{x})\\
&= 0
\end{aligned}$$

$$\text{i.e., } (\widehat{x}-F) \wedge T^{***}(\widehat{x},\widehat{na}-\widehat{x}) = 0.$$

Since this holds for each n, we get

$$\sup_n((\widehat{x}-F) \wedge T^{***}(\widehat{x},\widehat{na}-\widehat{x})) = 0,$$

which leads that, by the separate order continuity of T^{***},

$$\begin{aligned}
0 \le (\widehat{x}-F) \wedge T^{***}(\widehat{x}-F,F) &= (\widehat{x}-F) \wedge T^{***}(\widehat{x},\sup_n(\widehat{na} \wedge \widehat{x}))\\
&= \sup_n((\widehat{x}-F) \wedge T^{***}(\widehat{x},\sup_n(\widehat{na} \wedge \widehat{x})))\\
&= 0;
\end{aligned}$$

that is,

$$(\widehat{x}-F) \wedge T^{***}(\widehat{x}-F,F) = 0.$$

Step 2. Let $F = \bigwedge_{i=1}^{m} F_i$ where either $F_i \in \mathcal{G}\widehat{a}$ or $\widehat{x}-F_i \in \mathcal{G}\widehat{a}$. Then

$$\widehat{x}-F = \bigvee_{i=1}^{m}(\widehat{x}-F_i),$$

and so

$$\begin{aligned}
0 \le (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) &= \bigvee_{i=1}^{m} (\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, \bigwedge_{i=1}^{m} F_i) \\
&\le \sum_{i=1}^{m} (\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, \bigwedge_{i=1}^{m} F_i) \\
&\le \sum_{i=1}^{m} (\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, \sum_{i=1}^{m} F_i) \\
&= \sum_{i=1}^{m} (\widehat{x} - F_i) \wedge \sum_{i=1}^{m} T^{***}(\widehat{x}, F_i) \\
&\le \sum_{i=1}^{m} ((\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, F_i)) \\
&= 0 \quad \text{(by Step 1)};
\end{aligned}$$

$$\text{i.e., } (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) = 0.$$

Step 3. Let $F = \bigvee_{i=1}^{n} F_i$ where each F_i is of the form F had in Step 1 (that is, $F_i = \bigwedge_{j=1}^{m} F_{ij}, \forall i = 1, 2, \cdots, n$, and so $F = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} F_{ij}$). Then, in the same way as Step 2,

$$\widehat{x} - F = \bigwedge_{i=1}^{m} (\widehat{x} - F_i),$$

and so

$$\begin{aligned}
0 \le (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) &= \bigwedge_{i=1}^{m} (\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, \bigvee_{i=1}^{m} F_i) \\
&\le \sum_{i=1}^{m} (\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, \bigvee_{i=1}^{m} F_i) \\
&\le \sum_{i=1}^{m} (\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, \sum_{i=1}^{m} F_i) \\
&= \sum_{i=1}^{m} (\widehat{x} - F_i) \wedge \sum_{i=1}^{m} T^{***}(\widehat{x}, F_i) \\
&\le \sum_{i=1}^{m} ((\widehat{x} - F_i) \wedge T^{***}(\widehat{x}, F_i)) \\
&= 0 \quad \text{(by Step 2)};
\end{aligned}$$

$$\text{i.e., } (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) = 0.$$

Step 4. Let $F \in \mathcal{R}\widehat{x}$, where $\mathcal{R}$ is as before. If $F = \sup_\alpha F_\alpha$ or $F = \inf_\alpha F_\alpha$ with each F_α is a component of $\widehat{x}$ (that is, $(\widehat{x} - F_\alpha) \wedge F_\alpha = 0$ for each α) having the property that $(\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\alpha) = 0$, then using the separate order continuity of T^{***} we show that F has the same property;

$$\text{i.e., } (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) = 0.$$

as follows.

Suppose that $F = \sup_\alpha F_\alpha$. Then it follows from

$$T^{***}(\widehat{x} - F) = T^{***}(\widehat{x}, F_\alpha)$$

for each $x \in A^+$ that

$$\begin{aligned} 0 \leq (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) &= \sup_\alpha((\widehat{x} - F) \wedge T^{***}(\widehat{x}, F_\alpha)) \\ &\leq \sup_\alpha((\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\alpha)) \\ &= 0 \quad \text{(by hypothesis);} \end{aligned}$$

$$\text{i.e., } (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) = 0.$$

We now show that if $F = \inf_\alpha F_\alpha$ such that $(\widehat{x} - F_\alpha) \wedge F_\alpha = 0$ and $(\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\alpha) = 0$ for each α, then

$$\text{i.e., } (\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) = 0.$$

For some fixed α and for all $\beta \geq \alpha$ we have

$$0 \leq (\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\beta) \leq (\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\beta) = 0.$$

Hence for all $\beta \geq \alpha$

$$(\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\beta) = 0,$$

and so

$$\inf_{\beta \geq \alpha}((\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F_\beta)) = 0.$$

By the separate order continuity of T^{***},

$$(\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, \inf_{\beta \geq \alpha} F_\beta) = 0,$$

$$\text{i.e., } (\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F) = 0.$$

Since this is true for all α,

$$\sup_\alpha((\widehat{x} - F_\alpha) \wedge T^{***}(\widehat{x}, F)) = 0.$$

It follows from

$$(\widehat{x} - \inf_\alpha F_\alpha) \wedge T^{***}(\widehat{x}, F) = 0,$$

that

$$(\widehat{x} - F) \wedge T^{***}(\widehat{x}, F) = 0,$$

which completes the proof. □

Next we aim to extend the above result; that is, if A is a vector lattice and $T : A \times A \to A$ is a bi-orthomorphism, then the triadjoint $T^{***} : A'' \times A'' \to A''$ is also a bi-orthomorphism.

Theorem 3.4. *Let A be a vector lattice and $T : A \times A \to A$ be a bi-orthomorphism. Then the bilinear map $T^{***} : A'' \times A'' \to A''$ is a bi-orthomorphism. In other words, if $T \in Orth(A, A)$, then $T^{***} \in Orth(A'', A'')$.*

Proof. Suppose that $T : A \times A \to A$ is a bi-orthomorphism. We first observe that, since every bi-orthomorphism is orthosymmetric by Lemma 3.2, all results in the previous section are true for the bi-orthomorphism T. We also recall that the triadjoint $T^{***} : A'' \times A'' \to A''$ is separately order bounded whenever T is a bi-orthomorphism. Now let $G \wedge H = 0$ in A'' and $0 \leq F \in A''$. Then we have

$$F = F_n + F_s, \quad G = G_n + G_s \quad \text{and} \quad H = H_n + H_s$$

with $0 \leq F_n, G_n, H_n \in (A')'_n$ and $0 \leq F_s, G_s, H_s \in (A')'_s$ as $A'' = (A')'_n \oplus (A')'_s$. It follows that, by Corollary 2.10, $T^{***}(F_n, G_n) \in (A')'_n$, and so

$$T^{***}(F_n, G_n) \wedge H_s = 0$$

as $0 \leq H_s \in (A')'_s$. On the other hand, $G_n \wedge H_n = 0$ since $G \wedge H = 0$, which gives that

$$T^{***}(F_n, G_n) \wedge H_n = 0,$$

as the map $T^{***} : (A')'_n \times (A')'_n \to (A')'_n$ is a bi-orthomorphism by Theorem 3.3. Therefore we have

$$\begin{aligned} 0 \leq T^{***}(F, G) \wedge H &= T^{***}(F_n, G_n) \wedge H_n + H_s \\ &\leq T^{***}(F_n, G_n) \wedge H_n + T^{***}(F_n, G_n) \wedge H_s \\ &= 0; \end{aligned}$$

$$\text{i.e., } T^{***}(F, G) \wedge H = 0,$$

as required. □

We conclude our work with the following consequences.

Corollary 3.5. *The order bidual A'' of an Archimedean almost f-algebra (respectively f-algebra) A is a Dedekind complete almost f-algebra (respectively f-algebra).*

Proof. We recall that the order bidual of any Archimedean lattice ordered algebra is a Dedekind complete lattice ordered algebra, equipped with the Arens multiplication [2, 3]. It is not difficult to see that the map $T^{***} : A'' \times A'' \to A''$ defined by

$$T^{***}(F, G) = F \cdot G \qquad (F, G \in A'')$$

is positive orthosymmetric by Theorem 2.9. Hence if $F \wedge G = 0$ in A'', then $F \cdot G = T^{***}(F, G) = 0$. □

Corollary 3.6. *Every Archimedean almost f-algebra (and so f-algebra) A is commutative.*

Proof. The map $T : A \times A \to A$ defined by

$$T(x, y) = xy \qquad (x, y \in A)$$

is positive orthosymmetric, and so symmetric since every positive orthosymmetric bilinear map is symmetric, as observed before. Hence

$$xy = T(x, y) = T(y, x) = yx$$

for all $x, y \in A$, as required. □

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