ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 6, 2013

EIGENVALUES AND EIGENFUNCTIONS OF DISCONTINUOUS TWO-POINT BOUNDARY VALUE PROBLEMS WITH AN EIGENPARAMETER IN THE BOUNDARY CONDITION

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ABSTRACT. In this paper we get asymptotic formulas for eigenvalues and eigenfunctions of discontinuous two-point boundary value problems with the eigenparameter in the boundary conditions with transmission conditions at the two points of discontinuity.

When our problem is continuous the obtained results coincide with the corresponding results in [**3**].

1. Introduction.

Consider

$$\tau u := -u'' + q(x)u = \lambda u \tag{1}$$

on $[-1, h_1) \bigcup (h_1, h_2) \bigcup (h_2, 1)$, with the boundary conditions

$$L_1(u) := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0 \tag{2}$$

$$L_2(u) := (\beta_1'\lambda + \beta_1)u(1) - (\beta_2'\lambda + \beta_2)u'(1) = 0 \tag{3}$$

and the transmission conditions

$$L_3(u) := u(h_1 - 0) - \delta u(h_1 + 0) = 0, \tag{4}$$

$$L_4(u) := u'(h_1 - 0) - \delta u'(h_1 + 0) = 0, \tag{5}$$

$$L_5(u) := u(h_2 - 0) - \gamma u(h_2 + 0) = 0, \tag{6}$$

$$L_6(u) := u'(h_2 - 0) - \gamma u'(h_2 + 0) = 0, \tag{7}$$

where $-1 < h_1 < h_2 < 1$, $q(x)$ is a given real-valued function continuous on $[-1, h_1), (h_1, h_2)$, $(h_2, 1]$ and has finite limits $q(h_i \pm 0) :=$

2010 AMS *Mathematics subject classification.* Primary 34L10, 47E05.
Keywords and phrases. Two-point boundary value problems, transmission condition, asymptotic of eigenvalues and eigenfunctions, discontinuous boundary-value problems.
Received by the editors on November 6, 2008, and in revised form on September 27, 2010.
DOI:10.1216/RMJ-2013-43-6-1871

$\lim_{x\longrightarrow h_i\pm 0} q(x)$ $(i = 1, 2)$; $\delta, \gamma, \alpha_i, \beta_i, \beta_i'$ $(i = 1, 2)$ are real numbers $|\alpha_1| + |\alpha_2| \neq 0$, $\delta \neq 0$, $\gamma \neq 0$; λ is the complex eigenvalue parameter. As for [**3**], we assume that $\rho = \beta_1'\beta_2 - \beta_2'\beta_1 > 0$.

This type of problem is considered in the references [**1–5, 7, 10, 11**].

Note that, as a rule, problems of this type arise in the theory of heat and mass transfer, in diffraction and in various processes of physical transfer problems.

2. Preliminaries. In the Hilbert space $H = L_2(-1, 1) \bigoplus C$, we define an inner product by

$$(F, G) := \int_{-1}^{h_1} f(x)\overline{g(x)}\,dx + \delta^2 \int_{h_1}^{h_2} f(x)\overline{g(x)}\,dx + \delta^2\gamma^2 \int_{h_2}^{1} f(x)\overline{g(x)}\,dx + \frac{\delta^2\gamma^2}{\rho} f_1\overline{g_1}$$

for

$$F := \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}, \qquad G := \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in H.$$

Following [**3**], for convenience we put

$$R_1(u) := \beta_1 u(1) - \beta_2 u'(1), \qquad R_1'(u) := \beta_1' u(1) - \beta_2' u'(1).$$

For functionals $f(x)$, which are defined on $[-1, h_1) \cup (h_1, h_2) \cup (h_2, 1]$ and have finite limits $f(h_i \pm 0) := \lim_{x\to h_i\pm 0}$ $(i = 1, 2)$. By $f_1(x)$, $f_2(x)$ and $f_3(x)$, we denote the functions

$$f_1(x) = \begin{cases} f(x) & x \in [-1, h_1) \\ \lim\limits_{x\to h_1-0} f(x) & x = h_1, \end{cases}$$

$$f_2(x) = \begin{cases} \lim\limits_{x\to h_1+0} f(x) & x = h_1 \\ f(x) & x \in (h_1, h_2)\ , \\ \lim\limits_{x\to h_2-0} f(x) & x = h_2 \end{cases}$$

$$f_3(x) = \begin{cases} \lim\limits_{x\to h_2+0} f(x) & x = h_2 \\ f(x) & x \in (h_2, 1] \end{cases}$$

which are defined on $\Omega_1 = [-1, h_1]$, $\Omega_2 = [h_1, h_2]$ and $\Omega_3 = [h_2, 1]$, respectively.

The operator-theoretic formulation of (1)–(7) assumes the form

$$AF = \begin{pmatrix} \tau f \\ -R_1(f) \end{pmatrix}$$

with $D(A) = \{F \in H \big| f_i(x), f_i'(x)$ are absolutely continuous in Ω_i $(i = 1, 2, 3)$, $\tau f \in L_2[-1, 1]$, $L_1 f = L_3 f = L_4 f = L_5 f = L_6 f = 0$ and $f_1 = R_1'(f)\}$. Consequently, the problem (1)–(7) can be considered as the eigenvalue problem for the operator A.

Theorem 2.1. *The operator A is symmetric.*

Proof. Let F, $G \in D(A)$. Integrating by parts twice, we get

$$\begin{aligned}\langle AF, G\rangle &= \langle F, AG\rangle + W(f, \overline{g}; h_1 - 0) - W(f, \overline{g}; -1)\\ &\quad + \delta^2 W(f, g; h_2 - 0) - \delta^2 W(f, g; h_1 + 0)\\ &\quad + \gamma^2\delta^2 W(f, g; 1) - \gamma^2\delta^2 W(f, g; h_2 + 0)\\ &\quad + \frac{\gamma^2\delta^2}{\rho}\big(R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g})\big),\end{aligned}$$

where

$$W(f, g; x) = f(x)g'(x) - f'(x)g(x)$$

denotes the Wronskian of the functions f and g. Since f and $\overline{g}$ satisfy boundary condition (2), it follows that

$$W(f, \overline{g}; -1) = 0. \tag{9}$$

From the transmission conditions (4)–(7), we get

$$W(f, g; h_1 - 0) = \delta^2 W(f, g; h_1 + 0) \tag{10}$$

$$W(f, g; h_2 - 0) = \gamma^2 W(f, g; h_2 + 0). \tag{11}$$

Further,

$$\begin{aligned}R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g})\\ &= \big(\beta_1' f(1) - \beta_2' f'(1)\big)\big(\beta_1\overline{g}(1) - \beta_2\overline{g}'(1)\big)\\ &\quad - \big(\beta_1 f(1) - \beta_2 f'(1)\big)\big(\beta_1'\overline{g}(1) - \beta_2'\overline{g}(1)\big)\\ &= (\beta_2\beta_1' - \beta_2'\beta_1)f'(1)\overline{g}(1) + (\beta_1\beta_2' - \beta_1'\beta_2)f(1)\overline{g}'(1)\\ &= \rho(f'(1)\overline{g}(1) - f(1)\overline{g}'(1)) = -\rho W(f, g, 1).\end{aligned} \tag{12}$$

Finally, substituting (9)–(12) in (8), we then get

$$\langle AF, G\rangle = \langle F, AG\rangle \quad (F, G \in D(A)). \tag{13}$$

Theorem 2.1 is proved. □

Corollary 2.1. *All eigenvalues of the problem* (1)–(7) *are real.*

We can now assume that all eigenfunctions are real valued.

Corollary 2.2. *If* λ_1 *and* λ_2 *are two different eigenvalues of the problem* (1)–(7)*, then the corresponding eigenfunctions* u_1 *and* u_2 *of this problem satisfy the following equality*

$$\int_{-1}^{h_1} u_1(x)u_2(x)\,dx + \delta^2 \int_{h_1}^{h_2} u_1(x)u_2(x)\,dx + \delta^2\gamma^2 \int_{h_2}^{1} u_1(x)u_2(x)\,dx = -\frac{\delta^2\gamma^2}{\rho} R_1'(u_1)R_1'(u_2).$$

In fact, this formula implies the orthogonality of the eigenfunctions u_1 *and* u_2 *in the Hilbert space* H.

We need the following lemma, which can be proved by the same technique as in the proof of Theorem 1.5 in [**7**].

Lemma 2.1. *Let the real-valued function* $q(x)$ *be continuous in* $[a, b]$*, and let* $f(\lambda), g(\lambda)$ *be given entire functions. Then, for any* $\lambda \in C$*, the equation*

$$-u'' + q(x)u = \lambda u, \quad x \in [a, b],$$

has a unique solution $u = u(x, \lambda)$ *satisfying the initial conditions*

$$u(a) = f(\lambda) \quad u'(a) = g(\lambda) \quad (\text{or } u(b) = f(\lambda) \quad u'(b) = g(\lambda)).$$

For each fixed $x \in [a, b]$*,* $u(x, \lambda)$ *is an entire function of* λ.

We shall define two solutions $\varphi_\lambda(x)$ and $\chi_\lambda(x)$ of equation (1) as follows.

Let $\varphi_{1\lambda}(x) := \varphi_1(x,\lambda)$ be the solution of equation (1) on $[-1,h_1]$ which satisfies the initial conditions

$$u(-1) = \alpha_2, \qquad u'(-1) = -\alpha_1. \tag{14}$$

By virtue of Lemma 2.1 after defining the above solution we may define the solution $\varphi_{2\lambda}(x) := \varphi_2(x,\lambda)$ of equation (1) on $[h_1,h_2]$ by means of the solution $\varphi_1(x,\lambda)$ by the initial conditions.

$$u(h_1) = \delta^{-1}\varphi_1(h_1,\lambda) \quad u'(h_1) = \delta^{-1}\varphi_1'(h_1,\lambda). \tag{15}$$

Again after defining this solution we may define the solution $\varphi_{3\lambda}(x) := \varphi_3(x,\lambda)$ of equation (1) on $[h_2,1]$ by means of $\varphi_2(x,\lambda)$ by the initial conditions.

$$u(h_2) = \gamma^{-1}\varphi_2(h_2,\lambda), \qquad u'(h_2) = \gamma^{-1}\varphi_2'(h_2,\lambda). \tag{16}$$

Hence,

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda) & x \in [-1,h_1) \\ \varphi_2(x,\lambda) & x \in (h_1,h_2) \\ \varphi_3(x,\lambda) & x \in (h_2,1] \end{cases}$$

satisfies equation (1) on $[-1,h_1)\bigcup(h_1,h_2)\bigcup(h_2,1]$, the boundary conditions (2) and the transmission conditions (4)–(7).

Similarly, first we define the solution $\chi_{3\lambda}(x) := \chi_3(x,\lambda)$ on $[h_2,1]$ by the initial conditions

$$u(1) = \beta_2'\lambda + \beta_2, \qquad u'(1) = \beta_1'\lambda + \beta_1. \tag{17}$$

After defining the above solution $\chi_{2\lambda}(x) = \chi_2(x,\lambda)$ of equation (1) on $[h_1,h_2]$, the initial conditions

$$u(h_2) = \gamma\chi_3(h_2,\lambda), \qquad u'(h_2) = \gamma\chi_3'(h_2,\lambda). \tag{18}$$

Again, after defining this solution, we define the solution $\chi_{1\lambda}(x) = \chi_1(x,\lambda)$ of equation (1) on $[-1,h_1]$ by the initial conditions

$$u(h_1) = \delta\chi_2(h_1,\lambda), \qquad u'(h_1) = \delta\chi_2'(h_1,\lambda). \tag{19}$$

Hence,

$$\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda) & x \in [-1,h_1) \\ \chi_2(x,\lambda) & x \in (h_1,h_2) \\ \chi_3(x,\lambda) & x \in (h_2,1] \end{cases}$$

satisfies equation (1) on $[-1,h_1)\bigcup(h_1,h_2)\bigcup(h_2,1]$, the boundary condition (3) and the transmissions (4)–(7).

It is obvious that the Wronskians

$$\begin{aligned}\omega_i(\lambda) &= W_\lambda(\varphi_i,\chi_i;x) \\ &= \varphi_i(x,\lambda)\chi_i'(x,\lambda) - \varphi_i'(x,\lambda)\chi_i(x,\lambda), \quad x \in \Omega_i \ (i=1,2,3)\end{aligned}$$

are independent of $x \in \Omega_i$ and entire functions.

Lemma 2.2. *For each $\lambda \in C$, $\omega_1(\lambda) = \delta^2\omega_2(\lambda) = \gamma^2\delta^2\omega_3(\lambda)$.*

Proof. Because of (15), (16), (18) and (19), a short calculation gives

$$\begin{aligned}W_\lambda(\varphi_1,\chi_1;h_1) &= \delta^2 W_\lambda(\varphi_2,\chi_2;h_1) \\ &= \delta^2 W_\lambda(\varphi_2,\chi_2;h_2) \\ &= \delta^2\gamma^2 W_\lambda(\varphi_3,\chi_3;h_2),\end{aligned}$$

so

$$\omega_1(\lambda) = \delta^2\omega_2(\lambda) = \delta^2\gamma^2\omega_3(\lambda).$$

Now we may introduce the characteristic function

$$\omega(\lambda) := \omega_1(\lambda) = \delta^2\omega_2(\lambda) = \delta^2\gamma^2\omega_3(\lambda).$$

Theorem 2.2. *The eigenvalues of the problem* (1)–(7) *are the zeros of the function $\omega(\lambda)$.*

Proof. Let $\omega(\lambda_0) = 0$. Then $W_{\lambda_0}(\varphi_1,\chi_1;x) = 0$, and therefore the functions $\varphi_1(x,\lambda)$ and $\chi_1(x,\lambda)$ are linearly independent, i.e.,

$$\chi_1(x,\lambda_0) = k_1\varphi_1(x,\lambda_0), \quad x \in [-1,h_1]$$

for some $k_1 \neq 0$. From this, it follows that $\chi_1(x,\lambda_0)$ satisfies the boundary condition (2) so $\chi_1(x,\lambda_0)$ is an eigenfunction of the problem (1)–(7) corresponding to this eigenvalue λ_0.

Now let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0, but $\omega(\lambda_0) \neq 0$. Then the functions $\varphi_1, \chi_1; \varphi_2, \chi_2$ and φ_3, χ_3 would be linearly dependent on $[-1, h_1]$, $[h_1, h_2]$ and $[h_2, 1]$, respectively. Therefore, $u_0(x)$ may be represented in the following form:

$$u_0(x) = \begin{cases} C_1\varphi_1(x,\lambda_0) + C_2\chi_1(x,\lambda_0) & x \in [-1, h_1) \\ C_3\varphi_2(x,\lambda_0) + C_4\chi_2(x,\lambda_0) & x \in (h_1, h_2) \\ C_5\varphi_3(x,\lambda_0) + C_6\chi_3(x,\lambda_0) & x \in (h_2, 1], \end{cases}$$

where at least one of the constants $C_1, C_2, C_3, C_4, C_5, C_6$ is not zero.

Considering the equations

$$L_\nu(u_0(x)) = 0, \quad \nu = 1, \dots, 6, \tag{20}$$

as a homogenous system of linear equations of the variables C_i ($i = 1, \dots, 6$) and taking into account (15), (16), (18), (19), it follows that the determinant of this system satisfies

$$\begin{vmatrix} 0 & \omega_1(\lambda_0) & 0 & 0 & 0 & 0 \\ \varphi_{1\lambda_0}(h_1) & \chi_{1\lambda_0}(h_1) & -\delta\varphi_{2\lambda_0}(h_1) & -\delta\chi_{2\lambda_0}(h_1) & 0 & 0 \\ \varphi'_{1\lambda_0}(h_1) & \chi'_{1\lambda_0}(h_1) & -\delta\varphi'_{2\lambda_0}(h_1) & -\delta\chi'_{2\lambda_0}(h_1) & 0 & 0 \\ 0 & 0 & \varphi_2(h_2) & \chi_{2\lambda_0}(h_2) & -\gamma\varphi_3(h_2) & -\gamma\chi_{3\lambda_0}(h_2) \\ 0 & 0 & \varphi'_2(h_2) & \chi'_{2\lambda_0}(h_2) & -\gamma\varphi'_3(h_2) & -\gamma\chi'_{3\lambda_0}(h_2) \\ 0 & 0 & 0 & 0 & \omega_3(\lambda_0) & 0 \end{vmatrix}$$
$$= -\delta^2\gamma^2\omega_1(\lambda_0)\omega_2(\lambda_0)\omega_3^2(\lambda_0).$$

Therefore, the system (20) has the only trivial solution $C_i = 0$ ($i = 1, \dots, 6$). Thus we get a contradiction, which completes the proof. □

Lemma 2.3. *If $\lambda = \lambda_0$ is an eigenvalue, then $\varphi(x,\lambda_0)$ and $\chi(x,\lambda_0)$ are linearly independent.*

Proof. Let $\lambda = \lambda_0$ be an eigenvalue. Then, by virtue of Theorem 2.2,

$$W(\varphi_i(x,\lambda_0), \chi(x,\lambda_0)) = \omega_i(\lambda_0) = 0,$$

and therefore,

(21) $$\chi_i(x,\lambda_0) = k_i\varphi_i(x,\lambda_0), \quad (i = 1, 2, 3),$$

for some $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$. We must show that $k_1 = k_2 = k_3$. Suppose, if possible, that $k_2 \neq k_3$; taking into account the definitions of the solution $\varphi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ from the equalities (21), we have

$$\begin{aligned} L_5(\chi_{\lambda_0}) &= \chi_{\lambda_0}(h_2 - 0) - \gamma\chi_{\lambda_0}(h_2 + 0) \\ &= \chi_{2\lambda_0}(h_2) - \gamma\chi_{3\lambda_0}(h_2) \\ &= k_2\varphi_2(h_2) - \gamma k_3\varphi_3(h_2) \\ &= k_2\gamma\varphi_3(h_2) - \gamma k_3\varphi_3(h_2) \\ &= \gamma(k_2 - k_3)\varphi_3(h_2) = 0. \end{aligned}$$

Since $L_5(\chi_{\lambda_0}) = 0$ and $\gamma(k_2 - k_3) \neq 0$, it follows that

(22) $$\varphi_{3\lambda_0}(h_2) = 0.$$

By the same procedure, from $L_6(\chi_{\lambda_0}) = 0$, we can derive that

(23) $$\varphi'_{3\lambda_0}(h_2) = 0.$$

From the fact that $\varphi_{3\lambda_0}(x)$ is a solution of differential equation (1) on $[h_2, 1]$ and satisfies the initial conditions (22) and (23), it follows that $\varphi_{3\lambda_0}(x) = 0$ identically on $[h_2, 1]$ because of the well-known existence and uniqueness theorem for the initial value problems of the ordinary linear differential equations. Making use of (16), (21) and (22), we may also find

(24) $$\varphi_{2\lambda_0}(h_2) = \varphi'_{2\lambda_0}(h_2) = 0.$$

From the latter discussion for $\varphi_{3\lambda_0}(x)$, it follows that $\varphi_{2\lambda_0}(x) = 0$ identically on $[h_1, h_2]$. Making use of (15) and (24), we may also find

$$\varphi_{1\lambda_0}(h_1) = \varphi'_{1\lambda_0}(h_1) = 0.$$

It follows that $\varphi_{1\lambda_0}(x) = 0$ identically on $[-1, h_1]$. Hence, $\varphi(x,\lambda_0) = 0$ identically on $[-1, h_1) \bigcup (h_1, h_2) \bigcup (h_2, 1]$. But this is a contradiction of (14). Hence, $k_2 = k_3$. Analogously, we can prove that $k_1 = k_2$.

Corollary 2.3. *If $\lambda = \lambda_0$ is an eigenvalue, then both $\varphi(x, \lambda_0) = 0$ and $\chi(x, \lambda_0) = 0$ are eigenfunctions corresponding to this eigenvalue.*

Lemma 2.4. *All eigenvalues λ_n are simple zeros of $\omega(\lambda)$.*

Proof. Using the well-known Lagrange's formula it can be shown that

$$(25)\quad (\lambda - \lambda_n)\bigg(\int_{-1}^{h_1} \varphi_\lambda(x)\varphi_{\lambda_0}(x)\,dx + \delta^2 \int_{h_1}^{h_2} \varphi_\lambda(x)\varphi_{\lambda_0}(x)\,dx + \delta^2\gamma^2 \int_{h_2}^{1} \varphi_\lambda(x)\varphi_{\lambda_0}(x)\,dx\bigg) = \delta^2\gamma^2 W(\varphi_\lambda, \varphi_{\lambda_0}; 1),$$

for any λ. Since $\chi_{\lambda_n}(x) = k_n \varphi_{\lambda_n}(x)$, $x \in [-1, h_1) \bigcup (h_1, h_2) \bigcup (h_2, 1]$ for some $k_n \neq 0$, $n = 1, 2, \dots$. Using this equality for the right side of (25), we have

$$\begin{aligned} W(\varphi_\lambda, \varphi_{\lambda_n}; 1) &= \frac{1}{k_n} W(\varphi_\lambda, \chi_{\lambda_n}; 1) \\ &= \frac{1}{k_n}\big(\lambda_n R_1'(\varphi_\lambda) + R_1(\varphi_\lambda)\big) \\ &= \frac{1}{k_n}\big[\omega(\lambda) - (\lambda - \lambda_n) R_1'(\varphi_\lambda)\big] \\ &= (\lambda - \lambda_n)\frac{1}{k_n}\bigg[\frac{\omega(\lambda)}{\lambda - \lambda_n} - R_1'(\varphi_\lambda)\bigg]. \end{aligned}$$

Substituting this formula in (25) and letting $\lambda \to \lambda_0$, we get

$$(26)\quad \int_{-1}^{h_1} \big(\varphi_{\lambda_n}(x)\big)^2 dx + \delta^2 \int_{h_1}^{h_2} \big(\varphi_{\lambda_n}(x)\big)^2 dx + \delta^2\gamma^2 \int_{h_2}^{1} \big(\varphi_{\lambda_n}(x)\big)^2 dx = \frac{\delta^2\gamma^2}{k_n}\omega'(\lambda_n) - \frac{\delta^2\gamma^2}{k_n} R_1'(\varphi_{\lambda_n}).$$

Now, putting

$$R_1'(\varphi_{\lambda_n}) = \frac{1}{k_n} R_1'(\chi_{\lambda_n}) = \frac{\rho}{k_n}$$

in (26), we get $\omega'(\lambda_n) \neq 0$.

3. Asymptotic approximate formulas for $\omega(\lambda)$. We begin by proving some lemmas.

Lemma 3.1. *Let $\varphi(x,\lambda)$ be the solution of equation* (1) *defined in Section* 2 *and* $\lambda = s^2$. *Then*

(27)
$$\frac{d^k\varphi_{1\lambda}(x)}{dx^k} = \alpha_2 \frac{d^k}{dx^k}\cos[s(x+1)] - \frac{\alpha_1}{s}\frac{d^k}{dx^k}\sin[s(x+1)] + \frac{1}{s}\int_{-1}^{x}\frac{d^k}{dx^k}\sin[s(x-y)]q(y)\varphi_{1\lambda}(y)\,dy, \quad k=0,1,$$

(28)
$$\frac{d^k\varphi_{2\lambda}(x)}{dx^k} = \frac{1}{\delta}\varphi_{1\lambda}(h_1)\frac{d^k}{dx^k}\cos[s(x-h_1)] + \frac{1}{s}\frac{1}{\delta}\varphi'_{1\lambda}(h_1)\frac{d^k}{dx^k}\sin[s(x-h_1)] + \frac{1}{s}\int_{h_1}^{x}\frac{d^k}{dx^k}\sin[s(x-y)]q(y)\varphi_{2\lambda}(y)\,dy, \quad k=0,1,$$

(29)
$$\frac{d^k\varphi_{3\lambda}(x)}{dx^k} = \frac{1}{\gamma}\varphi_{2\lambda}(h_2)\frac{d^k}{dx^k}\cos[s(x-h_2)] + \frac{1}{s}\frac{1}{\gamma}\varphi'_{2\lambda}(h_2)\frac{d^k}{dx^k}\sin[s(x-h_2)] + \frac{1}{s}\int_{h_2}^{x}\frac{d^k}{dx^k}\sin[s(x-y)]q(y)\varphi_{3\lambda}(y)\,dy, \quad k=0,1.$$

Proof. We just use $s^2\varphi_{1\lambda}(y) + \varphi''_{1\lambda}(y)$, $s^2\varphi_{2\lambda}(y) + \varphi''_{2\lambda}(y)$ and $s^2\varphi_{3\lambda}(y) + \varphi''_{3\lambda}(y)$ instead of $q(y)\varphi_{1\lambda}(y)$, $g(y)\varphi_{2\lambda}(y)$ and $q(y)\varphi_{3\lambda}(y)$ in the integral term of (27), (28) and (29), respectively, and integrate by parts twice.

Lemma 3.2. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *Then the functions* $\varphi_{i\lambda}(x)$ *have the following asymptotic representation for* $|\lambda| \to \infty$, *which holds* *uniformly for* $x \in \Omega_i$ (*for* $i = 1, 2, 3$)

(30)
$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = \alpha_2 \frac{d^k}{dx^k}\cos[s(x+1)] + O\big(|s|^{k-1}e^{|t|(x+1)}\big)$$

(31)
$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = \frac{\alpha_2}{\delta}\frac{d^k}{dx^k}\cos[s(x+1)] + O\big(|s|^{k-1}e^{|t|(x+1)}\big)$$

(32)
$$\frac{d^k}{dx^k}\varphi_{3\lambda}(x) = \frac{\alpha_2}{\delta\gamma}\frac{d^k}{dx^k}\cos[s(x+1)] + O\big(|s|^{k-1}e^{|t|(x+1)}\big)$$

if $\alpha_2 \neq 0$.

(33)
$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -\frac{\alpha_1}{s}\frac{d^k}{dx^k}\sin[s(x+1)] + O\big(|s|^{k-2}e^{|t|(x+1)}\big)$$

(34)
$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = -\frac{\alpha_1}{\delta s}\frac{d^k}{dx^k}\sin[s(x+1)] + O\big(|s|^{k-2}e^{|t|(x+1)}\big)$$

(35)
$$\frac{d^k}{dx^k}\varphi_{3\lambda}(x) = -\frac{\alpha_1}{\delta\gamma s}\frac{d^k}{dx^k}\sin[s(x+1)] + O\big(|s|^{k-2}e^{|t|(x+1)}\big)$$

if $\alpha_2 = 0$.

Proof. Since the proofs of the formula for $\varphi_{1\lambda}(x)$ are identical to Titchmarsh's proof of similar results for $\varphi_\lambda(x)$ (see [**8**, Lemma 1.7, pages 9–10]), we may formulate them without proof. But similar formulas for $\varphi_{2\lambda}(x)$ and $\varphi_{3\lambda}(x)$ need individual consideration, since the last solutions are defined by initial conditions of special nonstandard forms. Therefore, we shall only prove formulas (31) and (32) (since (34) and (35) may be proved similarly to (31) and (32)).

Let $\alpha_2 \neq 0$. It follows from (30) that

(36)
$$\varphi_{1\lambda}(h_1) = \alpha_2 \cos[s(h_1+1)] + O\big(|s|^{-1}e^{|t|(h_1+1)}\big)$$

and

$$\varphi'_{1\lambda}(h_1) = -s\alpha_2 \sin[s(h_1+1)] + O\big(e^{|t|(h_1+1)}\big). \tag{37}$$

Putting (36) and (37) in (28) (for $k = 0$), we get

$$\begin{aligned}
(38)\quad & \varphi_{2\lambda}(x) \\
&= \frac{1}{\delta}\Big(\alpha_2 \cos[s(h_1+1)] + O\big(|s|^{-1}e^{|t|(h_1+1)}\big)\Big)\cos[s(x-h_1)] \\
&\quad + \frac{1}{s\delta}\Big(-s\alpha_2 \sin[s(h_1+1)] + O\big(e^{|t|(h_1+1)}\big)\Big)\sin[s(x-h_1)] \\
&\quad + \frac{1}{s}\int_{h_1}^{x} \sin[s(x-y)]q(y)\varphi_{2\lambda}(y)\,dy \\
&= \frac{\alpha_2}{\delta}\Big(\cos[s(h_1+1)]\cos[s(x-h_1)] - \sin[s(h_1+1)]\sin[s(x-h_1)]\Big) \\
&\quad + O\left(\frac{e^{|t|(x+1)}}{|s|}\right) + \frac{1}{s}\int_{h_1}^{x} \sin[s(x-y)]q(y)\varphi_{2\lambda}(y)\,dy \\
&= \frac{\alpha_2}{\delta}\cos[s(x+1)] + \frac{1}{s}\int_{h_1}^{x} \sin[s(x-y)]q(y)\varphi_{2\lambda}(y)\,dy \\
&\quad + O\left(\frac{e^{|t|(x+1)}}{|s|}\right).
\end{aligned}$$

Denoting $F_{2\lambda}(x) = e^{-|t|(x+1)}\varphi_{2\lambda}(x)$ from (38) we have

$$\begin{aligned}
F_{2\lambda}(x) = {} & \frac{\alpha_2}{\delta} e^{-|t|(x+1)} \cos[s(x+1)] \\
& + \frac{1}{s}\int_{h_1}^{x} \sin[s(x-y)]q(y)e^{-|t|(x-y)}F_{2\lambda}(y)\,dy \\
& + O\left(\frac{1}{|s|}\right).
\end{aligned}$$

Denoting $M_1(\lambda) = \max_{h_1 \le x \le h_2} |F_{2\lambda}(x)|$, it follows that

$$M_1(\lambda) \le \frac{|\alpha_2|}{\delta} + \frac{M_1(\lambda)}{|s|}\int_{h_1}^{h_2} |q(y)|\,dy + \frac{M_0}{|s|}$$

for some $M_0 > 0$. From this, it follows that $M_1(\lambda) = O(1)$ as $\lambda \to \infty$. So

$$\varphi_{2\lambda}(x) = O\big(e^{|t|(x+1)}\big).$$

Substituting in the integral on the right of (38), we have

$$\varphi_{2\lambda}(x) = \frac{\alpha_2}{\delta}\cos[s(x+1)] + O\left(\frac{e^{|t|(x+1)}}{|s|}\right).$$

So formula (31) follows for $k = 0$. Similarly, putting (36) and (37) in (30) for $k = 1$ and following the same technique, we can get formula (31) for $k = 1$,

$$\varphi'_{2\lambda}(x) = -\frac{\alpha_2 s}{\delta}\sin[s(x+1)] + O\big(e^{|t|(x+1)}\big).$$

Now we prove formula (32) for $k = 0$. It follows from (31) that

$$\varphi_{2\lambda}(h_2) = \frac{\alpha_2}{\delta}\cos[s(h_2+1)] + O\left(\frac{e^{|t|(h_2+1)}}{|s|}\right) \tag{39}$$

$$\varphi'_{2\lambda}(h_2) = -s\frac{\alpha_2}{\delta}\sin[s(h_2+1)] + O\big(e^{|t|(h_2+1)}\big). \tag{40}$$

Putting (39) and (40) in (29) (for $k = 0$), we have

(41)

$$\begin{aligned}\varphi_3(x) &= \frac{1}{\gamma}\left(\frac{\alpha_2}{\delta}\cos[s(h_2+1)]\cos[s(x-h_2)]\right)\\ &\quad + \frac{1}{s\gamma}(-s)\left(\frac{\alpha_2}{\delta}\sin[s(h_2+1)]\sin[s(x-h_2)]\right)\\ &\quad + \frac{1}{s}\int_{h_2}^{x}\sin[s(x-y)]q(y)\varphi_3(y)\,dy + O\left(\frac{e^{|t|(x+1)}}{|s|}\right)\\ &= \frac{\alpha_2}{\gamma\delta}\cos[s(x+1)] + \frac{1}{s}\int_{h_2}^{x}\sin[s(x-y)]q(y)\varphi_3(y)\,dy\\ &\quad + O\left(\frac{e^{|t|(x+1)}}{|s|}\right).\end{aligned}$$

Denoting $F_{3\lambda}(x) = e^{-|t|(x+1)}\varphi_{3\lambda}(x)$, from (41) we have

$$\begin{aligned}F_{3\lambda}(x) &= \frac{\alpha_2}{\gamma\delta}e^{-|t|(x+1)}\cos[s(x+1)]\\ &\quad + \frac{1}{s}\int_{h_2}^{x}\sin[s(x-y)]q(y)e^{-|t|(x-y)}F_{3\lambda}(y)\,dy + O\left(\frac{1}{|s|}\right).\end{aligned}$$

Denoting $M_2(\lambda) = \max_{h_2 \le x \le 1} |F_{3\lambda}(x)|$ from the last formula, it follows that

$$M_2(\lambda) \le \left|\frac{\alpha_2}{\gamma\delta}\right| + \frac{M_2(\lambda)}{|s|}\int_{h_2}^{1} |q(y)|\,dy + \frac{M_3}{|s|}$$

for some $M_3 > 0$. From this, it follows that $M_2(\lambda) = O(1)$ as $\lambda \to \infty$. So

$$\varphi_{3\lambda}(x) = O\big(e^{|t|(x+1)}\big).$$

Substituting in the integral on the right of (41), we have

$$\varphi_{3\lambda}(x) = \frac{\alpha_2}{\gamma\delta}\cos[s(x+1)] + O\bigg(\frac{e^{|t|(x+1)}}{s}\bigg).$$

So formula (32) follows for $k = 0$. The other assertions can be proved similarly. □

Theorem 3.1. *Let $\lambda = s^2$, $t = \operatorname{Im} s$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic representations.*

Case 1. $\beta_2' \neq 0$, $\alpha_2 \neq 0$.

$$\omega(\lambda) = \alpha_2\beta_2'\delta\gamma s^3 \sin 2s + O\big(|s|^2 e^{2|t|}\big). \tag{42}$$

Case 2. $\beta_2' \neq 0$, $\alpha_2 = 0$.

$$\omega(\lambda) = \beta_2'\alpha_1\delta\gamma s^2 \cos 2s + O\big(|s| e^{2|t|}\big). \tag{43}$$

Case 3. $\beta_2'' = 0$, $\alpha_2 \neq 0$.

$$\omega(\lambda) = \beta_1'\alpha_2\delta\gamma s^2 \cos 2s + O\big(|s| e^{2|t|}\big). \tag{44}$$

Case 4. $\beta_2' = 0$, $\alpha_2 = 0$.

$$\omega(\lambda) = -\beta_1'\alpha_1\delta\gamma s \sin 2s + O\big(e^{2|t|}\big). \tag{45}$$

Proof. The proof is completed by substituting (32) and (35) in the representation:

$$\begin{aligned}\omega(\lambda) &= \delta^2\gamma^2\omega_3(\lambda) = \delta^2\gamma^2[\varphi_{3\lambda}(1)\chi_{3\lambda}(1) - \varphi_{3\lambda}'(1)\chi_{3\lambda}'(1)]\\ &= \delta^2\gamma^2[(\lambda\beta_1' + \beta_1)\varphi_{3\lambda}(1) - (\lambda\beta_2' + \beta_2)\varphi_{3\lambda}'(1)]\\ &= \lambda\delta^2\gamma^2(\beta_1'\varphi_{3\lambda}(1) - \beta_2'\varphi_{3\lambda}'(1))\\ &\quad + \delta^2\gamma^2(\beta_1\varphi_{3\lambda}(1) - \beta_2\varphi_{3\lambda}'(1)).\end{aligned} \tag{46}$$

□

Corollary 3.1. *The eigenvalues of the problem* (1)–(7) *are bounded below.*

Proof. Putting $s = it$ $(t > 0)$ in the above formulas, it follows that $\omega(-t^2) \to \infty$ as $t \to \infty$. Hence, $\omega(\lambda) \neq 0$ for λ negative and sufficiently large in module.

4. Asymptotic formulas for eigenvalues and eigenfunctions. In this section we shall obtain the asymptotic approximation formula for the eigenvalues of the considered problem (1)–(7).

Since the eigenvalues coincide with zeros of the entire function $\omega(\lambda)$, it follows that they have no finite limit. Moreover, it is clear from Corollary 2.1 and Corollary 3.1 that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ listed according to their multiplicities.

Theorem 4.1. *The eigenvalues* $\lambda_n = s_n^2$, $n = 0, 1, 2, \ldots,$ *of the problem* (1)–(7) *have the following asymptotic representation for* $n \to \infty$.

Case 1. $\beta_2' \neq 0$, $\alpha_2 \neq 0$.

$$s_n = \frac{(n-1)\pi}{2} + O\left(\frac{1}{n}\right). \tag{47}$$

Case 2. $\beta_2' \neq 0$, $\alpha_2 = 0$.

$$s_n = \frac{(n-(1/2))\pi}{2} + O\left(\frac{1}{n}\right). \tag{48}$$

Case 3. $\beta_2' = 0$, $\alpha_2 \neq 0$.

$$s_n = \frac{(n-(1/2))\pi}{2} + O\left(\frac{1}{n}\right). \tag{49}$$

Case 4. $\beta_2' = 0$, $\alpha_2 = 0$.

$$s_n = \frac{\pi n}{2} + O\left(\frac{1}{n}\right). \tag{50}$$

Proof. We shall only consider the first case (the other cases are similar).

Denoting by $\omega_1(s)$ and $\omega_2(s)$, the first and the O-term of the right of (42), respectively, we shall apply the well-known Rouche theorem which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour C, and $|g(s)| < |f(s)|$ on C, then $f(s)$ and $f(s) + g(s)$ have the same number of zeros inside C, provided that each zero is counted according to its multiplicity.

We now show that$|\omega_1(s)| > |\omega_2(s)|$ on the contours

$$C_n := \left\{ s \in C \quad |s| = \frac{(n + (1/2))\pi}{2} \right\}$$

for sufficiently large n.

Let $\lambda_0 \leq \lambda_1 \leq \cdots$ be the zeros $\omega(\lambda)$ and $\lambda_n = s_n^2$. Since, inside the contour C_n, $\omega_1(s)$ has zero at the points $s = 0$ (with multiplicity 4) and $s = (\pi k)/2$, $k = \pm 1, \pm 2, \ldots, \pm n$ (with multiplicity 1) and so the number of zero is $2n + 4$, it follows that

$$s_n = \frac{(n-1)\pi}{2} + \delta_n \tag{51}$$

where $\delta_n = O(1)$, more precisely $|\delta_n| < (\pi/4)$ for sufficiently large n. By substituting this in (42), we derive that $\delta_n = O(1/n)$, which completes the proof.

The next approximation for the eigenvalues may be obtained by the following procedure. For this, we shall suppose that $q(y)$ is of bounded variation on $[-1, 1]$.

We only consider the case $\beta_2' \neq 0$, $\alpha_2 \neq 0$ (since other cases may be considered similarly).

Putting $x = h_2$ in (28) and then substituting in (29), we get

$$\begin{aligned}\varphi_{3\lambda}'(1) = -\frac{s}{\gamma}\bigg(&\frac{1}{\delta}\varphi_{1\lambda}(h_1)\cos[s(h_2 - h_1)] \\ &+ \frac{1}{s\delta}\varphi_{1\lambda}''(h_1)\sin[s(h_2 - h_1)] \\ &+ \frac{1}{s}\int_{h_1}^{h_2} \sin[s(h_2 - y)]q(y)\varphi_{2\lambda}(y)\,dy\bigg)\sin[s(1 - h_2)]\end{aligned}$$

$$+\frac{1}{\gamma}\bigg(-\frac{s}{\delta}\varphi_1(h_1)\sin[s(h_2-h_1)]+\frac{1}{\delta}\varphi_1'(h_1)\cos[s(h_2-h_1)]$$
$$+\int_{h_1}^{h_2}\cos[s(h_2-y)]q(y)\varphi_{2\lambda}(y)\,dy\bigg)\cos[s(1-h_2)]$$
$$+\int_{h_2}^{1}\cos[s(1-y)]g(y)\varphi_{3\lambda}(y)\,dy$$
$$=-\frac{s}{\gamma\delta}\varphi_1(h_1)\sin[s(1-h_1)]+\frac{1}{\gamma\delta}\varphi_1'(h_1)\cos[s(1-h_1)]$$
$$+\frac{1}{\gamma}\int_{h_1}^{h_2}\cos[s(1-y)]q(y)\varphi_{2\lambda}(y)\,dy$$
$$+\int_{h_2}^{1}\cos[s(1-y)]q(y)\varphi_{3\lambda}(y)\,dy.$$

Putting $x=h_1$ in (27) and then substituting in the last equality, we get

$$\varphi_{3\lambda}'(1)=-\frac{s}{\gamma\delta}\bigg(\alpha_2\cos[s(h_1+1)]-\frac{\alpha_1}{s}\sin[s(h_1+1)]$$
$$+\frac{1}{s}\int_{-1}^{h_1}\sin[s(h_1-y)]q(y)\varphi_{1y}(y)\,dy\bigg)\sin[s(1-h_1)]$$
$$+\frac{1}{\gamma\delta}\bigg(-\alpha_2 s\sin[s(h_1+1)]-\alpha_1\cos[s(h_1+1)]$$
$$+\int_{-1}^{h_1}\cos[s(h_1-y)]q(y)\varphi_{1\lambda}(y)\,dy\bigg)\cos[s(1-h_1)]$$
$$+\frac{1}{\gamma}\int_{h_1}^{h_2}\cos[s(1-y)]q(y)\varphi_2(y)\,dy$$
$$+\int_{h_2}^{1}\cos[s(1-y)]q(y)\varphi_{3\lambda}(y)\,dy$$
$$=-\frac{\alpha_2}{\gamma\delta}s\sin 2s-\frac{\alpha_1}{\gamma\delta}\cos 2s$$
$$+\frac{1}{\gamma\delta}\int_{-1}^{h_1}\cos[s(1-y)]q(y)\varphi_{1\lambda}(y)\,dy$$
$$+\frac{1}{\gamma}\int_{h_1}^{h_2}\cos[s(1-y)]q(y)\varphi_2(y)\,dy$$

$$+\int_{h_2}^{1}\cos[s(1-y)]q(y)\varphi_3(y)\,dy.$$

Substituting (30), (31) and (32) on the right side of the last integral equality then gives

$$\begin{aligned}\varphi'_{3\lambda}(1) &= -\frac{\alpha_2}{\gamma\delta}s\sin 2s-\frac{\alpha_1}{\delta\gamma}\cos 2s\\ &\quad+\frac{\alpha_2}{\delta\gamma}\int_{-1}^{h_1}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy\\ &\quad+\frac{\alpha_2}{\delta\gamma}\int_{h_1}^{h_2}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy\\ &\quad+\frac{\alpha_2}{\delta\gamma}\int_{h_2}^{1}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy\\ &\quad+O\big(|s|^{-1}e^{2|t|}\big)\\ &= -\frac{\alpha_2}{\gamma\delta}s\sin 2s-\frac{\alpha_1}{\delta\gamma}\cos 2s\\ &\quad+\frac{\alpha_2}{\delta\gamma}\int_{-1}^{1}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy\\ &\quad+O\big(|s|^{-1}e^{2|t|}\big).\end{aligned}$$

On the other hand, from (32) it follows that

$$\varphi_{3\lambda}(1)=\frac{\alpha_2}{\delta\gamma}\cos 2s+O\big(|s|^{-1}e^{2|t|}\big).$$

Putting these formulas in (46), we get

$$\begin{aligned}\omega(\lambda) &= s^2\delta^2\gamma^2\bigg[\beta'_1\bigg(\frac{\alpha_2}{\delta\gamma}\cos 2s+O\big(|s|^{-1}e^{2|t|}\big)\bigg)\\ &\qquad-\beta'_2\bigg(-\frac{\alpha_2}{\gamma\delta}s\sin 2s-\frac{\alpha_1}{\delta\gamma}\cos 2s\\ &\qquad\quad+\frac{\alpha_2}{\delta\gamma}\int_{-1}^{1}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy+O\big(|s|^{-1}e^{2|t|}\big)\bigg)\bigg]\\ &\qquad+\delta^2\gamma^2\bigg[\beta_1\bigg(\frac{\alpha_2}{\delta\gamma}\cos 2s+O\big(|s|^{-1}e^{2|t|}\big)\bigg)\end{aligned}$$

$$
\begin{aligned}
&- \beta_2\biggl(- \frac{\alpha_2}{\gamma\delta} s \sin 2s - \frac{\alpha_1}{\delta\gamma}\cos 2s \\
&+ \frac{\alpha_2}{\delta\gamma}\int_{-1}^{1}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy \\
&\qquad\qquad\qquad\qquad + O\bigl(|s|^{-1}e^{2|t|}\bigr)\biggr)\biggr] \\
= {}& s^3\alpha_2\beta_2'\delta\gamma\sin 2s + s^2\Bigl[\alpha_2\beta_1'\delta\gamma\cos 2s + \alpha_1\beta_2'\delta\gamma\cos 2s \\
&- \alpha_2\beta_2'\delta\gamma\int_{-1}^{1}\cos[s(1-y)]\cos[s(y+1)]q(y)\,dy\Bigr] + O\bigl(|s|e^{2|t|}\bigr) \\
= {}& s^3\alpha_2\beta_2'\delta\gamma\sin 2s + s^2\delta\gamma\Bigl[(\alpha_2\beta_1' + \alpha_1\beta_2')\cos 2s \\
&\qquad - \frac{1}{2}\alpha_2\beta_2'\cos 2s\int_{-1}^{1}q(y)\,dy - \frac{1}{2}\alpha_2\beta_2'\int_{-1}^{1}\cos 2sy\,q(y)\,dy\Bigr] \\
&+ O\bigl(|s|e^{2|t|}\bigr).
\end{aligned}
$$

We find, by putting (47) in the last equality,

$$
\begin{aligned}
\sin 2\delta_n = {}& \frac{\cos 2\delta_n}{s_n}\biggl[- \frac{\beta_1'}{\beta_2'} - \frac{\alpha_1}{\alpha_2} + \frac{1}{2\delta\gamma}\int_{-1}^{1}q(y)\,dy \\
&\qquad + \frac{1}{2\delta\gamma}\int_{-1}^{1}\cos(2s_n y)\,q(y)\,dy\biggr] + O\Bigl(\frac{1}{n^2}\Bigr).
\end{aligned}
\tag{52}
$$

By the Riemann-Lebesgue lemma, the second integral on the right in (52) is $O(n^{-1})$ of $q(x)$ bounded variation in $[-1,1]$ (see [**12**, page 48, Theorem 4.12]). Equation (52) therefore suggests that

$$
\delta_n = \frac{1}{(n-1)\pi}\biggl[- \frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2} + \frac{1}{2\delta\gamma}\int_{-1}^{1}q(y)\,dy\biggr] + O\Bigl(\frac{1}{n^2}\Bigr).
$$

Substituting in (47), we have

$$
s_n = \frac{(n-1)\pi}{2} + \frac{1}{(n-1)\pi}\biggl[- \frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2} + \frac{1}{2\delta\gamma}\int_{-1}^{1}q(y)\,dy\biggr] + O\Bigl(\frac{1}{n^2}\Bigr).
$$

Similar formulas in the other cases are as follows.

In Case 2,

$$s_n = \frac{(n-(1/2))\pi}{2} + \frac{1}{(n-(1/2))\pi}\left[\frac{\beta_1'}{\beta_2'} + \frac{1}{2\delta\gamma}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

In Case 3,

$$s_n = \frac{(n-(1/2))\pi}{2} + \frac{1}{(n-(1/2))\pi}\left[-\frac{\alpha_1}{\alpha_2} + \frac{\beta_1'}{\beta_2'} + \frac{1}{2\delta\gamma}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

In Case 4,

$$s_n = \frac{n\pi}{2} + \frac{1}{n\pi}\left[\frac{\beta_2}{\beta_1'} + \frac{1}{2\delta\gamma}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

Let $q(x,\lambda)$ be defined as in Section 2. Recalling that $\varphi(x,\lambda_n)$ is an eigenfunction according to the eigenvalue λ_n by putting (47) into equations (30), (31) and (32), we derive that

$$\varphi_{1\lambda_n}(x) = \alpha_2 \cos\left(\frac{(n-1)\pi(x+1)}{2}\right) + O\left(\frac{1}{n}\right)$$
$$\varphi_{2\lambda_n}(x) = \frac{\alpha_2}{\delta}\cos\left(\frac{(n-1)\pi(x+1)}{2}\right) + O\left(\frac{1}{n}\right)$$

and

$$\varphi_{3\lambda_n}(x) = \frac{\alpha_2}{\delta\gamma}\cos\left(\frac{(n-1)\pi(x+1)}{2}\right) + O\left(\frac{1}{n}\right).$$

Hence, if $\beta_2' \neq 0$ and $\alpha_2 \neq 0$, then the eigenfunction $\varphi(x,\lambda_n)$ has the asymptotic representation

$$\varphi(x,\lambda_n) = \begin{cases} \alpha_2\cos(\frac{(n-1)\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x\in[-1,h_1] \\ \frac{\alpha_2}{\delta}\cos(\frac{(n-1)\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x\in(h_1,h_2) \\ \frac{\alpha_2}{\delta\gamma}\cos(\frac{(n-1)\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x\in(h_2,1]. \end{cases}$$

Similar formulas in the other cases are as follows. In Case 2,

$$\varphi(x,\lambda_n) = \begin{cases} -\frac{2\alpha_1}{(n-(1/2))\pi}\sin(\frac{(n-(1/2))\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in [-1,h_1] \\ -\frac{2\alpha_1}{\delta(n-(1/2))\pi}\sin(\frac{(n-(1/2))\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in (h_1,h_2) \\ -\frac{2\alpha_1}{\delta\gamma(n-(1/2))\pi}\sin(\frac{(n-(1/2))\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in (h_2,1]. \end{cases}$$

In Case 3,

$$\varphi(x,\lambda_n) = \begin{cases} \alpha_2\cos(\frac{(n-(1/2))\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in [-1,h_1] \\ \frac{\alpha_2}{\delta}\cos(\frac{(n-(1/2))\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in (h_1,h_2) \\ \frac{\alpha_2}{\delta\gamma}\cos(\frac{(n-(1/2))\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in (h_2,1]. \end{cases}$$

In Case 4,

$$\varphi(x,\lambda_n) = \begin{cases} -\frac{2\alpha_1}{n\pi}\sin(\frac{n\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in [-1,h_1] \\ -\frac{1}{\delta}\frac{2\alpha_1}{n\pi}\sin(\frac{n\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in (h_1,h_2) \\ -\frac{1}{\delta\gamma}\frac{2\alpha_1}{n\pi}\sin(\frac{n\pi(x+1)}{2}) + O(\frac{1}{n}) & \text{for } x \in (h_2,1]. \end{cases}$$

All these asymptotic approximations hold uniformly for $x \in [-1,h_1) \bigcup (h_1,h_2) \bigcup (h_2,1]$.

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