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Full Length Research Paper

Family of surface with a common null geodesic

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We analyzed the problem of finding a surfaces family through a null curve with Cartan frame. We constructed a differential equation system such that a given null curve is an isogeodesic curve on a surface. In addition, developablity along the common null geodesic of the members of surface family are discussed. The extension to ruled surfaces is also outlined. Finally, we demonstrated some interesting surfaces about subject.

Key words: Cartan frame; surface family; common geodesic; ruled surface.

INTRODUCTION

Most people have heard the phrase; a straight line is the shortest distance between two points. But in differential geometry, they say this same thing in a different language. They say instead Geodesics for the Euclidean metric are straight lines. A geodesic is a curve that represents the extremal value of a distance function in some space. In the Euclidean space, extremal means 'minimal', so geodesics are paths of minimal arc length. In the 3-dimensional Minkowski space, the extremal paths are actually 'maximal' arc length.

In general relativity, geodesics generalize the notion of "straight lines" to curved space time. This concept is based on the mathematical concept of a geodesic. Importantly, the world line of a particle free from all external force is a particular type of geodesic. In other words, a freely moving particle always moves along a geodesic.

On a Minkowski surface, tangent vectors are classified into timelike, null, or space like, and so a (smooth) curve on the surface is said to be timelike, null, or space-like if its tangent vectors are always time-like, null, or space-like, respectively. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Null curves correspond to moving at the speed of light and space-like curves to moving faster than light.

Geodesics are curves along which geodesic curvature vanishes. This is of course where the geodesic curvature has its name from. Since Lorentzian metric is not positive definite metric, the distance function dS^2 can be positive, negative or zero, whereas the distance function in the Euclidean space can only be positive. Thus, we have to separate our geodesics on the basis of whether the distance function is positive, negative or zero. Geodesics with $dS^2 < 0$ are called spacelike geodesics. Geodesics with $dS^2 > 0$ are called timelike geodesics. Geodesics with $dS^2 = 0$ are called null geodesics. Note that a geodesic cannot be space-like at one point and timelike at another. Timelike geodesics correspond to freely-falling observers. Null geodesics correspond to light rays in the geometric optics approximation.

Surfaces with common geodesic in Minkowski 3-space have been the subject of many studies. Shirokov (1925) determined all two dimensional pseudo-Riemannian manifolds with common geodesics. The work of P. A. Shirokov laid the foundation for systematic research into geodesically corresponding spaces with non-definite metrics. Petrov (1949) used Shirokov's ideas to give a classification of geodesically corresponding pseudo-Riemannian spaces, and his student Golikov (1963) determined all four-dimensional Lorentz Spaces with corresponding geodesics. The classification of n-dimensional, geodesically correspond-ding, Lorentz spaces was completed by Kruchkovich (1963). Aminova (1993) studied solving the classic geometrical problem of determining the pseudo-Riemannian metrics that have corresponding geodesics. Kasap and Akyildiz (2006) constructed family of surfaces which is obtained a given spacelike (or timelike) geodesic curve. They derived the necessary and sufficient condi-tions for surfaces family with common spacelike (or timelike) geodesic. In this paper, we first take a null curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ on surface $\boldsymbol{\varphi} = \boldsymbol{\varphi}(s, v)$. Using Cartan frame of the curve, we

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derive the necessary and sufficient conditions such that the curve is an isogeodesic on the surface $\boldsymbol{\varphi}=\boldsymbol{\varphi}(s,v)$. Thus, we define the family of surfaces with common null geodesic. Moreover, we give the relation about developablity along the common null geodesic of the members of surfaces family. After we explain how to carry over these ideas to the case of ruled surfaces in the last section.

MATERIALS AND METHODS

Let IR_1^3 be Minkowski 3-space with natural Lorentzian metric $<.,.>=-dx^2+dy^2+dz^2$. A Cartan frame denoted by $\{\boldsymbol{l}(s),\boldsymbol{n}(s),\boldsymbol{u}(s)\}$ on a null curve $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ with tangent vector $\boldsymbol{l}=\boldsymbol{l}(s)$ where $<\boldsymbol{n},\boldsymbol{n}>=0$, $<\boldsymbol{l},\boldsymbol{n}>=-1$, $<\boldsymbol{l},\boldsymbol{u}>=0$, $<\boldsymbol{n},\boldsymbol{u}>=1$ and $\det(\boldsymbol{l},\boldsymbol{n},\boldsymbol{u})<0$ (Duggal and Bejancu, 1996).

The null frame $\{\boldsymbol{l}(s),\boldsymbol{n}(s),\boldsymbol{u}(s)\}$ satisfies the following Frenet -Serret formula:

$$\boldsymbol{l}'=k_1\boldsymbol{u},\quad \boldsymbol{n}'=-k_2\boldsymbol{u},\quad \boldsymbol{u}'=-k_2\boldsymbol{l}+k_1\boldsymbol{n} \tag{2.1}$$

Where $k_1=\left\|\boldsymbol{l}'\right\|$ and $k_2=-<\boldsymbol{n}',\boldsymbol{u}>$ are are called the curvature functions of $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ (Duggal and Bejancu, 1996). For the Cartan frame $\{\boldsymbol{l}(s),\boldsymbol{n}(s),\boldsymbol{u}(s)\}$, it is easy to see that:

$$\boldsymbol{l}\times\boldsymbol{n}=\boldsymbol{u},\quad \boldsymbol{n}\times\boldsymbol{u}=-\boldsymbol{n},\quad \boldsymbol{l}\times\boldsymbol{u}=\boldsymbol{l} \tag{2.2}$$

A curve on a surface is geodesic if and only if the normal vector to the curve is everywhere parallel to the local normal vector of the surface (Nassar et al., 2004).

An isoparametric curve $\boldsymbol{\alpha}(s)$ is a curve on a surface $\boldsymbol{\varphi}=\boldsymbol{\varphi}(s,v)$ in IR_1^3 that has a constant s or v-parameter value. In other words, there exists a parameter s_o or v_o such that $\boldsymbol{\alpha}(s)=\boldsymbol{\varphi}(s_o,v)$ or $\boldsymbol{\alpha}(s)=\boldsymbol{\varphi}(s,v_o)$. Given a parametric curve $\boldsymbol{\alpha}(s)$, we call $\boldsymbol{\alpha}(s)$ an isogeodesic of a surface $\boldsymbol{\varphi}$ if it is both a geodesic and an isoparametric curve on $\boldsymbol{\varphi}$.

Surfaces with common null geodesic

$\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ be an isoparametric null curve with Cartan frame $\{\boldsymbol{l},\boldsymbol{n},\boldsymbol{u}\}$ on surface $\boldsymbol{\varphi}=\boldsymbol{\varphi}(s,v)$. We assume that $\left\|\boldsymbol{\alpha}''(s)\right\|\neq 0$.

The surface $\boldsymbol{\varphi}$ is defined by

$$\boldsymbol{\varphi}(s,v)=\boldsymbol{\alpha}(s)+[x(s,v)\boldsymbol{l}(s)+y(s,v)\boldsymbol{n}(s)+z(s,v)\boldsymbol{u}(s)] \tag{3.1}$$

Where $x(s,v)$, $y(s,v)$ and $z(s,v)$ are C^1 functions. Our goal is to find the necessary and sufficient conditions for which the curve $\boldsymbol{\alpha}$ is a geodesic on the surface $\boldsymbol{\varphi}$. First, since $\boldsymbol{\alpha}$ is an isoparametric curve, we get

$$x(s,v_o)=y(s,v_o)=z(s,v_o)=0 \tag{3.2}$$

For the normal of $\boldsymbol{\varphi}=\boldsymbol{\varphi}(s,v)$, we can write

$$\boldsymbol{N}(s,v)=\frac{\partial\boldsymbol{\varphi}(s,v)}{\partial s}\times\frac{\partial\boldsymbol{\varphi}(s,v)}{\partial v}.$$

From (2.1) and (3.1), the normal vector can be expressed as:

$$\begin{aligned}\boldsymbol{N}(s,v)=&\left[\left(1+\frac{\partial x(s,v)}{\partial s}-z(s,v)k_2\right)\frac{\partial z(s,v)}{\partial v}-\left(\frac{\partial z(s,v)}{\partial s}+x(s,v)k_1-y(s,v)k_2\right)\frac{\partial x(s,v)}{\partial v}\right]\boldsymbol{l}+\\&\left[\left(\frac{\partial z(s,v)}{\partial s}+x(s,v)k_1-y(s,v)k_2\right)\frac{\partial y(s,v)}{\partial v}-\left(\frac{\partial y(s,v)}{\partial s}+z(s,v)k_1\right)\frac{\partial z(s,v)}{\partial v}\right]\boldsymbol{n}+\\&\left[\left(1+\frac{\partial x(s,v)}{\partial s}-z(s,v)k_2\right)\frac{\partial y(s,v)}{\partial v}-\left(\frac{\partial y(s,v)}{\partial s}+z(s,v)k_1\right)\frac{\partial x(s,v)}{\partial v}\right]\boldsymbol{u}.\end{aligned}$$

According to the geodesic theory (Nassar et al., 2004), Geodesic curvature $k_g=\det(\boldsymbol{l},\boldsymbol{l}',\boldsymbol{n})$ vanishes along geodesics. Thus, if we get:

$$\begin{aligned}\phi_1(s,v_o)&=\left(1+\frac{\partial x(s,v_o)}{\partial s}\right)\frac{\partial z(s,v_o)}{\partial v}-\frac{\partial z(s,v_o)}{\partial s}\frac{\partial x(s,v_o)}{\partial v},\\ \phi_2(s,v_o)&=\frac{\partial z(s,v_o)}{\partial s}\frac{\partial y(s,v_o)}{\partial v}-\frac{\partial y(s,v_o)}{\partial s}\frac{\partial z(s,v_o)}{\partial v},\\ \phi_3(u,v_o)&=\left(1+\frac{\partial x(s,v_o)}{\partial s}\right)\frac{\partial y(s,v_o)}{\partial v}-\frac{\partial y(s,v_o)}{\partial s}\frac{\partial x(s,v_o)}{\partial v}\end{aligned}$$

Then, we obtain

$$\phi_2(s,v_o)=0 \quad \text{Or} \quad k_2=0 \tag{3.3}$$

Since $\boldsymbol{\alpha}'$ is null vector and $<\boldsymbol{\alpha}',\boldsymbol{\alpha}''>=0$, $\boldsymbol{\alpha}''$ is a space-like or null vector. Because $\left\|\boldsymbol{\alpha}''\right\|\neq 0$, $\boldsymbol{\alpha}''$ is a space-like vector, thus, we can take $\boldsymbol{u}=\lambda\boldsymbol{\alpha}''$. From (3.3) we get:

$$\boldsymbol{N}(s,v_o)=\phi_1(s,v_o)\boldsymbol{\alpha}'+\phi_3(s,v_o)\lambda\boldsymbol{\alpha}''$$

We know that $\boldsymbol{\alpha}$ is a geodesic if and only if

$$\phi_1(s,v_o)=0 \text{ and } \phi_3(s,v_o)\neq 0. \tag{3.4}$$

Combining the conditions (3.2), (3.3) and (3.4), we have found the necessary and sufficient conditions for the surface $\boldsymbol{\varphi}$ to have the null curve $\boldsymbol{\alpha}$ as an isogeodesic. We call the set of surfaces defined by (3.1), (3.2), (3.3) and (3.4) the family of surfaces with common null geodesic. Any surface $\boldsymbol{\varphi}(s,v)$ defined by (3.1) and satisfying (3.2), (3.3) and (3.4) is a member of this family. The Gaussian curvature K of a surface in IR_1^3 is given by $K=\varepsilon\dfrac{h_{\alpha\beta}}{g_{\alpha\beta}}$, where $\varepsilon=<\boldsymbol{N},\boldsymbol{N}>$, and $g_{\alpha\beta}$, $h_{\alpha\beta}$ are first and second fundamental quantities, respectively (Tsagas and Papanyoniou, 1988). Thus, the surface $\boldsymbol{\varphi}$ is developable along the common null geodesic $\boldsymbol{\alpha}$ if and only if;

$$<\boldsymbol{\varphi}_{ss},\boldsymbol{\varphi}_s\times\boldsymbol{\varphi}_v>\big|_{v=v_o}<\boldsymbol{\varphi}_{vv},\boldsymbol{\varphi}_s\times\boldsymbol{\varphi}_v>\big|_{v=v_o}-(<\boldsymbol{\varphi}_{sv},\boldsymbol{\varphi}_s\times\boldsymbol{\varphi}_v>\big|_{v=v_o})^2=0$$

From (2.1), (3.1), (3.2) and (3.4), we can easily see that: The surface $\boldsymbol{\varphi}$ is developable along the common null geodesic $\boldsymbol{\alpha}$ if and only if;

$$\left[\left(1+\frac{\partial x}{\partial s}\right)k_1-\frac{\partial y}{\partial s}k_2+\frac{\partial^2 z}{\partial s^2}\right]\Bigg|_{v=v_o}\frac{\partial^2 z}{\partial v^2}-\left(\frac{\partial^2 s}{\partial s\partial v}\right)^2\Bigg|_{v=v_o}=0 \quad (3.5)$$

In equation (3.1), $x(s,v)$, $y(s,v)$ and $z(s,v)$ functions can be chosen in two different forms:

1.) If we choose:

$$x(s,v)=\sum_{i=1}^{p}a_{1i}k(s)^i x(v)^i,\quad y(s,v)=\sum_{i=1}^{p}a_{2i}m(s)^i y(v)^i$$

and $z(s,v)=\sum_{i=1}^{p}a_{3i}t(s)^i z(v)^i$ (3.6)

Then, we can simply express the sufficient condition for which the null curve $\boldsymbol{\alpha}$ is an isogeodesic on the surface $\boldsymbol{\varphi}$ as:

$$\begin{cases} x(v_o)=y(v_o)=z(v_o)=0 \\ a_{31}=0,\ \text{or}\ t(\text{s})=0,\ \text{or}\ \dfrac{dz}{dv}\Big|_{v=v_o}=0 \\ a_{21}\neq 0,\ m(s)\neq 0\ \text{and}\ \dfrac{dy}{dv}\Big|_{v=v_o}=\text{const}\neq 0 \end{cases} \quad (3.7)$$

Where $k(s),m(s),t(s),x(v),\ y(v)$ and $z(v)$ are C^1 functions, $a_{ij}\in IR$ $(i=1,2,3;\quad j=1,2,...,p)$ and $k(s),m(s)$ and $t(s)$ are not identically zero.

2.) If we get:

$$x(s,v)=f\left(\sum_{i=1}^{p}a_{1i}k(s)^i x(v)^i\right),$$

$$y(s,v)=g\left(\sum_{i=1}^{p}a_{2i}m(s)^i y(v)^i\right),$$

$$z(s,v)=h\left(\sum_{i=1}^{p}a_{3i}t(s)^i z(v)^i\right) \quad (3.8)$$

then, we can simply express the sufficient condition for which the curve $\boldsymbol{\alpha}$ is an isogeodesic on the surface $\boldsymbol{\varphi}$ as

$$\begin{cases} x(v_o)=y(v_o)=z(v_o)=0\ \text{and}\ f(0)=g(0)=h(0)=0 \\ a_{31}=0,\ \text{or t(s)}=0,\ \text{or}\ h'(0)=0,\ \text{or}\ \dfrac{dz}{dv}\Big|_{v=v_o}=0 \\ a_{21}\neq 0,\ \text{m}(s)\neq 0,\ g'(0)\neq 0\ \text{and}\ \dfrac{dy}{dv}\Big|_{v=v_o}=\text{const}\neq 0 \end{cases} \quad (3.9)$$

where $k(s),m(s),t(s),x(v),y(v),z(v),f,g$ and h are C^1 functions.

Example 1

Let $\alpha(s)=(s,\ sin(\text{s}),\ \cos(s))$ be a null curve. Then α is framed by;

$$\boldsymbol{l}(s)=(1,\ cos(\text{s}),-sin(\text{s})),$$

$$\boldsymbol{n}(s)=\left(\frac{1}{2},\ -\frac{1}{2}cos(\text{s}),\ \frac{1}{2}sin(\text{s})\right),$$

$$\boldsymbol{u}(s)=(0,\ -sin(\text{s}),\ -cos(\text{s})).$$

a.) If we take:

$$x(s,v)=\sum_{i=1}^{4}cos^i(s)sin^i(v),$$

$$y(s,v)=\sum_{i=1}^{4}cosh^i(s)sinh^i(v),$$

$$z(s,v)=0$$

and $v_o=0$ then (3.7) is satisfied. Thus, we immediately obtain a member of surfaces family with common null geodesic $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ (Figure 1) as:

$$\varphi_1(s,v)=\left(s+\sum_{i=1}^{4}\left(cos^i(s)sin^i(v)+\frac{1}{2}cosh^i(s)sinh^i(v)\right),\right.$$
$$sin(s)+\sum_{i=1}^{4}cos(s)\left(cos^i(s)sin^i(v)-\frac{1}{2}cosh^i(s)sinh^i(v)\right),$$
$$\left.cos(s)+\sum_{i=1}^{4}sin(s)\left(\frac{1}{2}cosh^i(s)sinh^i(v)-cos^i(s)sin^i(v)\right)\right).$$

where $-1\leq s\leq 0{,}8$ and $-1\leq v\leq 0{,}8$. From (3.5), The surface φ_1 is developable along common null geodesic $\boldsymbol{\alpha}$.

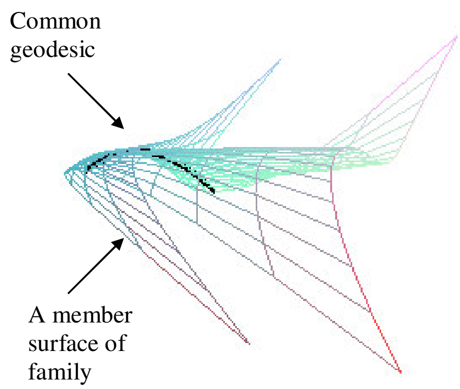

Figure 1. A developable member of surfaces family with common null geodesic.

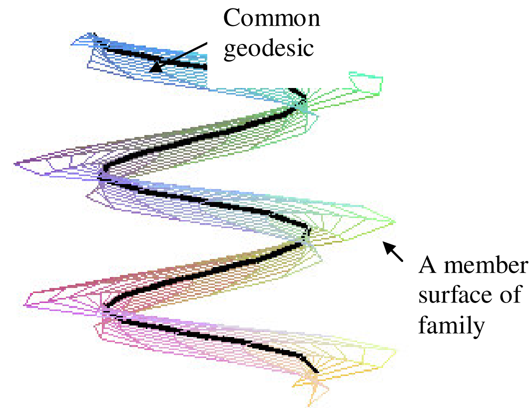

Figure 2. A developable member of surfaces family with common null geodesic.

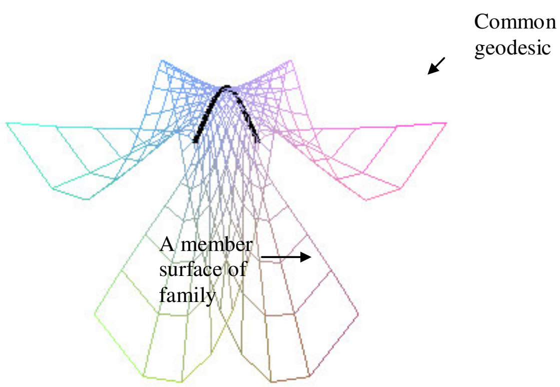

Figure 3. A non-developable member of surfaces family with common null geodesic.

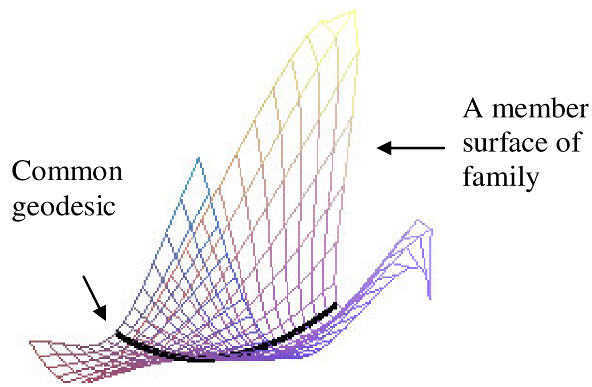

Figure 4. A non-developable member of surfaces family with common null geodesic.

b.) If we take, $x(s,v)=v^2,\ y(s,v)=v,\ z(s,v)=0$ and $v_o=0$ then we obtain another member of surfaces family with common null geodesic $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ as shown in Figure 2:

$$\varphi_2(s,v)=\left(s+v^2+\frac{1}{2}v,\ sin(s)+v^2cos(s)-\frac{1}{2}vcos(s),\ cos(s)-v^2sin(s)+\frac{1}{2}vsin(s)\right)$$

Where $-8\le s\le 8$ and $-1\le v\le 1$. From (3.5), The surface φ_2 is developable along common null geodesic $\boldsymbol{\alpha}$.

c.) If we take, $x(s,v)=0,\ y(s,v)=sinh(v),\ z(s,v)=sin(s^2v^2)$ and $v_o=0$ then we obtain another member of surfaces family with common null geodesic $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ as shown in Figure 3:

$$\varphi_3(s,v)=\left(s+\frac{1}{2}sinh(v),\ sin(s)-\frac{1}{2}cos(s)sinh(v)-s^4v^2sin(s),\ \frac{1}{2}sin(s)sinh(v)-s^4v^2cos(s)\right)$$

where $-1,5\le s\le 1,5$ and $-1,9\le v\le 1,9$. From (3.5), The surface φ_3 is not developable along common null geodesic $\boldsymbol{\alpha}$.

Example 2

Let $\alpha(s)=\left(sinh(s),\cosh(s),s\right)$ be a null curve. We will construct a surfaces family sharing the curve $\boldsymbol{\alpha}(s)$ as geodesic. It is easy to show that

$$\boldsymbol{l}(s)=\left(cosh(s),sinh(s),1\right),$$
$$\boldsymbol{n}(s)=\left(\frac{1}{2}cosh(s),\frac{1}{2}sinh(s),-\frac{1}{2}\right),$$
$$\boldsymbol{u}(s)=\left(sinh(s),\cosh(s),0\right).$$

If we choose; $x(s,v)=0,\ y(s,v)=sinh(v),\ z(s,v)=sin(s^2v^2)$, and $v_o=0$ then (3.9) is satisfied. Thus, we immediately obtain a member of surfaces family with common null geodesic $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$ (Figure 4) as:

$$\varphi_4(s,v)=\left(sinh(s)\left(1+sin(s^2v^2)\right)+\frac{1}{2}sinh(v)cosh(s),\ cosh(s)\left(1+sin(s^2v^2)\right)+\frac{1}{2}sinh(s)sinh(v),\ s-\frac{1}{2}sinh(v)\right)$$

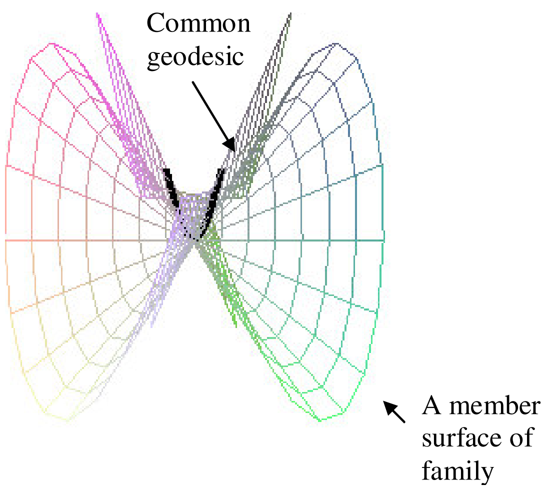

Figure 5. A member of ruled surfaces family with common null geodesic.

where $-0,7 \le s \le 1$ and $-1,5 \le v \le 1,5$. From (3.5), The surface φ_4 is not developable along common null geodesic $\boldsymbol{\alpha}$.

Ruled surfaces with common null geodesic

Ruled surfaces are one of the simplest objects in geometric modeling as they are generated basically by moving a line in space.

A surface $\boldsymbol{\varphi}$ is called a ruled surface in IR_1^3, if it is a surface swept out by a straight line l moving alone a curve $\boldsymbol{\alpha}$. The generating line l and the curve $\boldsymbol{\alpha}$ are called the rulings and the base curve of the surface, respectively.

We show how to derive the formulation of a ruled surfaces family such that the common null geodesic is also the base curve of ruled surfaces.

Let $\boldsymbol{\varphi} = \boldsymbol{\varphi}(s,v)$ be a ruled surface with the null isoparametric base curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. From the definition of ruled surface, there is a vector $\boldsymbol{R} = \boldsymbol{R}(s)$ such that;

$$\boldsymbol{\varphi}(s,v) - \boldsymbol{\varphi}(s,v_o) = (v - v_o)\boldsymbol{R}(s)$$

From (3.1), we get

$$(v - v_o)\boldsymbol{R}(s) = x(s,v)\boldsymbol{l}(s) + y(s,v)\boldsymbol{n}(s) + z(s,v)\boldsymbol{u}(s) \quad (4.1)$$

For the solutions of three unknowns $x(s,v)$, $y(s,v)$ and $z(s,v)$, we have;

$$\begin{aligned} x(s,v) &= -(v - v_o) < \boldsymbol{R}(s), \boldsymbol{n}(s) > \\ y(s,v) &= -(v - v_o) < \boldsymbol{R}(s), \boldsymbol{l}(s) > \\ z(s,v) &= \quad (v - v_o) < \boldsymbol{R}(s), \boldsymbol{u}(s) > \end{aligned} \quad (4.2)$$

If we take $\det(\boldsymbol{R}(s), \boldsymbol{l}(s), \boldsymbol{u}(s)) \neq 0$ and $\det(\boldsymbol{R}(s), \boldsymbol{l}(s), \boldsymbol{n}(s)) = 0$ then (3.2), (3.3) and (3.4) are satisfied. It follows that the at any point on the curve $\boldsymbol{\alpha}$, the vector $\boldsymbol{R}(s)$ must be in the plane formed by $\boldsymbol{l}(s)$ and $\boldsymbol{n}(s)$. On the other hand, the vectors $\boldsymbol{R}(s)$ and $\boldsymbol{l}(s)$ must not be parallel. This implies;

$$\boldsymbol{R}(s) = x(s)\boldsymbol{l}(s) + y(s)\boldsymbol{n}(s), \quad y(s) \neq 0 \quad (4.3)$$

So, the ruled surfaces family with common null geodesic is given by;

$$\boldsymbol{\varphi}(s,v) = \boldsymbol{\alpha}(s) + v x(s)\boldsymbol{l}(s) + v y(s)\boldsymbol{n}(s), \quad y(s) \neq 0 \quad (4.4)$$

From (3.5), all of the members of the ruled surfaces family with common null geodesic are developable.

Example 3.

Let $\alpha(s) = \left(-\frac{\sqrt{2}}{12}s^3 - \frac{\sqrt{2}}{2}s,\ -\frac{s^2}{2},\ -\frac{\sqrt{2}}{12}s^3 + \frac{\sqrt{2}}{2}s \right)$ be a null curve. The curve is called a null cubic. Then α is framed by:

$$\boldsymbol{l}(s) = \left(-\frac{\sqrt{2}}{4}s^2 - \frac{\sqrt{2}}{2},\ -s,\ -\frac{\sqrt{2}}{4}s^2 + \frac{\sqrt{2}}{2} \right),\ \boldsymbol{n}(s) = \left(-\frac{\sqrt{2}}{2},\ 0,\ -\frac{\sqrt{2}}{2} \right) \text{ and}$$

$$\boldsymbol{u}(s) = \left(-\frac{\sqrt{2}}{2}s,\ -1,\ -\frac{\sqrt{2}}{2}s \right).$$

If we take $x(s) = cos^2(s)$ and $y(s) = 1$ then (4.3) is satisfied. Thus, we immediately obtain a member of ruled surfaces family with common null geodesic $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ (Figure 5) as:

$$\varphi_5(s,v) = \left(-\frac{\sqrt{2}}{12}s^3 - \frac{\sqrt{2}}{2}(s+v) - v cos^2(s)\left(\frac{\sqrt{2}}{4}s^2 + \frac{\sqrt{2}}{2}\right),\ -\frac{s^2}{2} - v s cos^2(s),\right.$$
$$\left. -\frac{\sqrt{2}}{12}s^3 + \frac{\sqrt{2}}{2}(s-v) + v cos^2(s)\left(-\frac{\sqrt{2}}{4}s^2 + \frac{\sqrt{2}}{2}\right) \right)$$

where $-2 \le s \le 2$ and $-13 \le v \le 13$.

b) If we take, $x(s) = sin^2(2s)cos^2(s)$ and $y(s) = 1$ then we obtain another member of ruled surfaces family with common

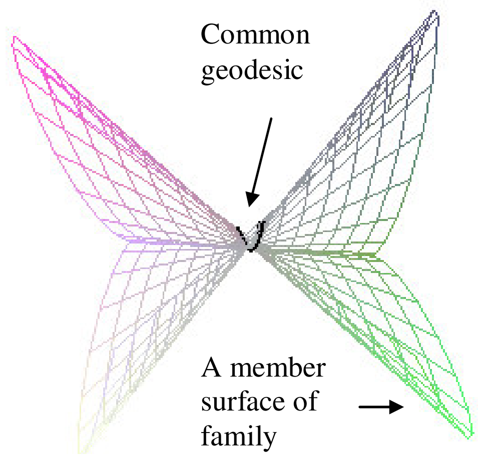

Figure 6. A member of ruled surfaces family with common null geodesic.

null geodesic $\alpha = \alpha(s)$ as shown in Figure 6:

$$\varphi_5(s,v) = \left(-\frac{\sqrt{2}}{12}s^3 - \frac{\sqrt{2}}{2}(s+v) - v\sin^2(2s)\cos^2(s)\left(\frac{\sqrt{2}}{4}s^2 + \frac{\sqrt{2}}{2}\right), -\frac{s^2}{2} - vs\sin^2(2s)\cos^2(s), \right.$$
$$\left. -\frac{\sqrt{2}}{12}s^3 + \frac{\sqrt{2}}{2}(s-v) + v\sin^2(2s)\cos^2(s)\left(-\frac{\sqrt{2}}{4}s^2 + \frac{\sqrt{2}}{2}\right) \right)$$

Where $-1 \le s \le 1$ and $-11 \le v \le 11$.

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