WONG-ZAKAI METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS IN ENGINEERING

by

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In this paper, Wong-Zakai approximation methods are presented for some stochastic differential equations in engineering sciences. Wong-Zakai approximate solutions of the equations are analyzed and the numerical results are compared with results from popular approximation schemes for stochastic differential equations such as Euler-Maruyama and Milstein methods. Several differential equations from engineering problems containing stochastic noise are investigated as numerical examples. Results show that Wong-Zakai method is a reliable tool for studying stochastic differential equations and can be used as an alternative for the known approximation techniques for stochastic models.

Key words: *Wong-Zakai approximation, stochastic differential equation, Euler-Maruyama method, Milstein method, solar irradiance, Brownian motion*

Introduction

Following the early studies on Brownian motion, including the renowned study of Albert Einstein in 1905, stochastic differential equations (SDE) have been used for modelling numerous events with fluctuating behavior. The state of the art of stochastic modelling stretches from financial applications and mathematical psychology to biological investigations and beyond. The use of stochastic noise in conjunction with deterministic differential equation systems as well as stochastic system models are popular applications in the field of epidemiology. Bacterial resistance [1], small cell lung cancer [2], and phytoplankton modelling [3] are only some of the stochastic modelling studies from natural sciences. In addition, there is also a vast literature of stochastic analysis in various branches of engineering. Many systems involved in energy, electrical, meteorological and marine engineering are known to be investigated via stochastic models.

In this study, SDE used in electric circuit analysis and solar irradiance predictions will be used as examples of such equations in engineering sciences and other applications [4-6]. Wong-Zakai approximations will be used to investigate these examples to present the advantages of Wong-Zakai schemes for analyzing stochastic systems in engineering problems. Although there are some probabilistic analyses as well, a considerable amount of the stochastic modelling studies involves the use of approximation methods. Euler-Maruyama and Milstein schemes are two of the most basic and popular tools used in the literature [7]. Similarly, various other deterministic schemes can be adapted to the stochastic mainframe through Ito's stochastic integration (along with other stochastic integrals) and the tools of stochastic calculus based

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on these operations, such as the stochastic Runge-Kutta method [8]. Stochastic finite element method [9] and random variable transformation method [10] are some of the other methods used in the literature. Models consisting of differential equations which are highly non-linear, coupled and of higher dimensions constitute a problem for the exact analysis of the solutions. Hence, a more accurate and/or efficient approximation method provides an alternative powerful tool for the analysis of such systems.

The leading studies of Wong and Zakai in 1965 [11, 12] presented an approximation to SDE with ODE. Wong-Zakai principle is based on the idea that the approximations of the noise in the SDE can be obtained by piecewise linear approximations of the Brownian motion [13]. Using this principle with Stratonovich stochastic integration, the common methods for ODE can be used for the investigation of stochastic models [14, 15]. Some of the current applications of the Wong-Zakai approximation in the literature are for analyzing SDE of higher dimension [16], SDE with reflecting boundary condition [17], SDE driven by martingales [18] and stochastic partial differential equations [19].

Considering that the use of noise terms for modelling physical systems is a popular application in mathematical modelling, we will compare Wong-Zakai approximate results with current popular stochastic schemes such as Euler-Maruyama, Milstein and Runge-Kutta. Methods for the solutions of ODE will be used for approximating the noise terms appearing in some engineering problems which will be used as examples to demonstrate the application of Wong-Zakai approximations in electrical engineering, energy engineering and similar fields. Our research suggests that Wong-Zakai approximation has some advantages in comparison these methods and can be very useful for analyzing SDE that arise in engineering problems as well as other branches of science.

Wong-Zakai approximation and methodology

In this section, Wong-Zakai approximation method is presented with the deterministic numerical approximation methods that will be used for analyzing SDE and the other stochastic methods that will be used for comparison.

Wong-Zakai Scheme

Wong-Zakai theorem is based on the approximation of SDE using ODE. Stratonovich integration plays a key point in this method, since unlike Ito stochastic integration, it abides the rules of ordinary calculus. A Stratonovich stochastic differential equation can be given:

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t)\circ \mathrm{d}W_t \tag{1}$$

with an initial condition $X_0 = x_0$ for $x_0 \in \mathbf{R}$. Note that we denote the stochastic process $X(t, \omega)$ by X_t. Here, the drift term $a(t, X_t)$ is sometimes denoted as $\underline{a}(t, X_t)$ to distinguish between Ito and Stratonovich stochastic integrals with the relation:

$$\underline{a}(t,x) = a(t,x) - \frac{1}{2} b(t,x)\frac{\partial}{\partial x} b(t,x) \tag{2}$$

As eq. (2) suggests, if the diffusion term $b(t, X_t)$ is independent of X_t, *i. e.* the SDE contains additive noise, Ito and Stratonovich SDE have the same drift coefficients. The interval $[0, T]$ is discretized as $0 = t_0 < t_1 < t_2 < ... < t_{k-1} < t_k = T$ to approximate the solution of (1) on these points. The method aims to find the numerical approximation $\hat{X}_{tj}$ of the solution X_{tj} for each interval $[t_j, t_{j+1}]$, $j = 0, 1,..., k - 1$ with the initial approximation $\hat{X}_0 = X_0 = x_0$. The $\hat{X}_{tj+1}$ is obtained by using the following initial value problem [15]:

$$\frac{\mathrm{d}\hat{X}_t}{\mathrm{d}t} = a\left[t, \hat{X}(t)\right] + \frac{1}{\Delta_j} b\left[t, \hat{X}(t)\right]\Delta W_j \tag{3}$$

for each sub-interval with $\hat{X}(t_j) = \hat{X}_j$. Here $\Delta_j = t_{j+1} - t_j$ and $\Delta W_j = W_{t_{j+1}} - W_{t_j}$ are the discrete approximations of dt and dW_t. Once Δ_j and ΔW_j are defined, schemes for ODE can be used with eq. (3) to obtain approximations of SDE on [0, T]. The deterministic multistep methods that will be used along with eq. (3) are Adams-Bashforth method and the predictor-corrector method with Adams-Bashforth as the predictor and Adams-Moulton as the corrector pair [20].

An illustration of how the deterministic schemes and Wong-Zakai are used together can be seen for a realization of an arbitrary stochastic process in the figure above, fig. 1.

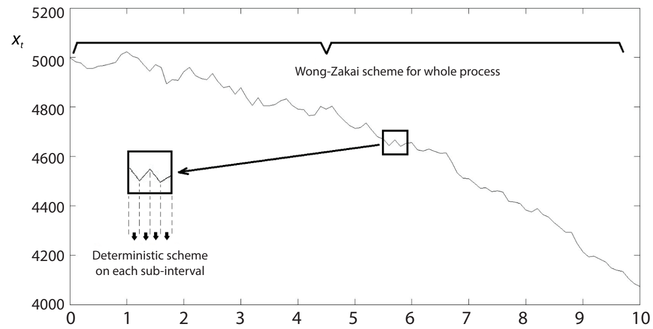

Figure 1. Wong Zakai approximation method for an arbitrary stochastic process

Stochastic schemes used for comparison

Several stochastic approximation methods have been used to validate the results obtained by Wong-Zakai method. The stochastic Runge-Kutta method is defined by using the Ito SDE:

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t \tag{4}$$

which can be turned into a Stratonovich SDE through eq. (2). The stochastic analogue of the fourth order Runge-Kutta scheme is given:

$$\bar{X}_{t_n} = \bar{X}_{t_{n-1}} + \frac{1}{6}\Big(\left[F_0 + 2F_1 + 2F_2 + F_3\right]h + \left[G_0 + 2G_1 + 2G_2 + G_3\right]\Delta W_n\Big) \tag{5}$$

where $h = t_n - t_{n-1}$ and

$$\begin{gathered} F_0 = a\left(t_{n-1}, \bar{X}_{t_{n-1}}\right),\ F_1 = a\left(t_{n-1} + \frac{1}{2}h, \bar{X}_{t_{n-1}} + \frac{1}{2}F_0 h + \frac{1}{2}G_0\Delta W_n\right) \\ F_2 = a\left(t_{n-1} + \frac{1}{2}h, \bar{X}_{t_{n-1}} + \frac{1}{2}F_1 h + \frac{1}{2}G_1\Delta W_n\right),\ F_3 = a\left(t_n, \bar{X}_{t_{n-1}} + F_2 h + G_2\Delta W_n\right) \end{gathered} \tag{6}$$

and G_i, $i = \overline{(1, 4)}$ are similarly obtained by the evaluation of the diffusion coefficient at the same point [8]. Note that here, $a(t, X_t)$ and $b(t, X_t)$ are the drift and diffusion coefficients of the Stratonovich SDE. The local equivalence between the stochastic fourth order Runge-Kutta method and the Milstein scheme in the mean square sense can be found in the literature [8]. The Milstein approximate solution X_t of the Ito SDE (4) on the interval [t_0, T], which is an order 1.0 strong Taylor scheme:

$$\bar{X}_{t_{n+1}} = \bar{X}_{t_n} + a\Delta_n + b\Delta W_n + \frac{1}{2}bb'\left[\left(\Delta W_n\right)^2 - \Delta_n\right] \tag{7}$$

for the discretized time interval $t_0 = \tau_0 < \tau_1 <\ldots < \tau_n < \tau_N = T$ with $n = 0, 1, 2,\ldots, N - 1$ using the initial value $\bar{X}_{t_0} = X_0$. Similarly for eq. (7), $\Delta_n = \tau_{n+1} - \tau_n$ and $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$. The strong order

1.0 of Milstein scheme is an improvement of the stochastic Euler scheme which is of strong order 0.5 [21]. Euler-Maruyama scheme is given for the discretized time interval and the initial value of the Ito SDE (4):

$$\overline{X}_{t_{n+1}} = \overline{X}_{t_n} + a\Delta_n + b\Delta W_n \tag{8}$$

where $\overline{X}_t$ is the approximate solution of the SDE. Both Euler and Milstein schemes have weak order 1.0 and for SDE with additive noise (7) and (8) are identical [21]. The explicit order 1.0 strong scheme, which is also called as Runge-Kutta in some resources, proposed by Platen is also used for comparison [7]:

$$Y_{n+1} = Y_n + a\Delta_n + b\Delta W_n + \frac{1}{2\sqrt{\Delta_n}}\left[b\left(\tau_n, \overline{\Upsilon}_n\right) - b\right]\left[\left(\Delta W_n\right)^2 - \Delta_n\right] \tag{9}$$

for $\overline{\Upsilon}_n = Y_n + a\Delta_n + b(\Delta_n)^{1/2}$ where Y is the approximate solution the Ito SDE.

Applications

The SDE used in modelling engineering problems are given as numerical examples to compare the efficiency of the method with other approximation methods. Note that to obtain the approximate expected values for the approximation methods, the mean of N simulated sample paths have been used:

$$E\left(X_t\right) \simeq \sum_{i=1}^{N} \frac{X_t^N}{N}$$

where X_t^N denotes the N^{th} sample path obtained for the method.

Problem 1. (A reducible SDE with non-linear multiplicative noise); consider the following SDE:

$$\mathrm{d}X_t = \beta^2 \sinh\left(X_t\right)\cosh^3\left(X_t\right)\mathrm{d}t + \beta\cosh^2\left(X_t\right)\mathrm{d}W_t \tag{10}$$

with the initial condition $X_0 = 1/2$. It is known that the exact solution of eq. (10) for $\beta = 1/10$ is obtained as [7]:

$$X_t = \operatorname{arctanh}\left[\tanh\left(\frac{1}{2}\right) + \frac{1}{10}W_t\right] \tag{11}$$

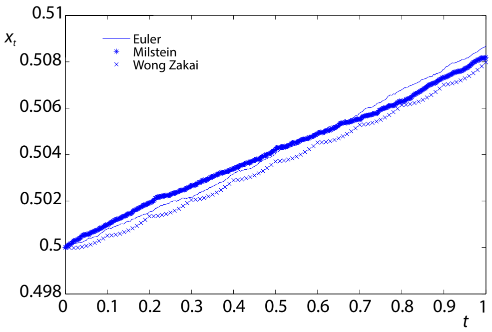

Figure 2. Approximate solutions of eq. (10)

The approximate solution of eq. (10) obtained with Euler, Miltein, and Wong-Zakai schemes are shown in the fig. 2. The Ito SDE (10) can be denoted as a Stratonovich SDE through the transformation (2):

$$\mathrm{d}X_t = \beta\cosh^2\left(X_t\right) \circ \mathrm{d}W_t \tag{12}$$

which will be used for Wong-Zakai method.

The results from the MATLAB-SDE algorithm, shown as SDE in the table, Euler-Maruyama scheme (8), Milstein scheme (7), Explicit order 1.0 Runge-Kutta (9), and stochastic fourth order Runge-Kutta scheme (5) are compared with Wong-Zakai scheme (3) together with deterministic Adams-Bashforth scheme and Wong-Zakai scheme (3) together with deterministic predictor-corrector scheme in the tab. 1. The approximate values have been obtained for a discretized sub-interval length $\Delta_n = 0.1$ for the stochastic methods and $h = 0.1$ for the deterministic methods used with Wong-Zakai method, tab. 1. The 10^6 realizations are used for the simulations of the expected values and we get the following results. The $\Delta_n = 0.01$ was used to obtain the results for Euler and Milstein methods to obtain the same number of total evaluation points with both Wong-Zakai methods.

Table 1. Comparison of results for eq. (10) using 10^6 simulations

t	Expected value, $E(X_t)$						SDE
	Stochastic schemes				Wong-Zakai scheme with		
	Euler--Maruyama	Milstein	Explicit	Runge-Kutta IV	Adams--Bashforth	Predictor--Corrector	
0.0	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.1	0.5007	0.5007	0.5007	0.5008	0.5008	0.5007	0.5009
0.2	0.5015	0.5015	0.5017	0.5015	0.5016	0.5015	0.5017
0.3	0.5023	0.5022	0.5023	0.5023	0.5024	0.5023	0.5025
0.4	0.5030	0.5029	0.5031	0.5032	0.5031	0.5031	0.5031
0.5	0.5038	0.5036	0.5039	0.5040	0.5039	0.5039	0.5042
0.6	0.5046	0.5045	0.5046	0.5048	0.5047	0.5047	0.5050
0.7	0.5055	0.5053	0.5056	0.5056	0.5055	0.5055	0.5059
0.8	0.5063	0.5062	0.5064	0.5065	0.5064	0.5063	0.5066
0.9	0.5072	0.5069	0.5071	0.5073	0.5072	0.5072	0.5074
1.0	0.5080	0.5078	0.5079	0.5082	0.5080	0.5082	0.5082

If relative error percentage (also known as percent error and denoted by δ) of the methods are compared (at $t = 1.0$ relative to the exact solution), the percentages of the absolute errors relative to the exact value are found:

$$\delta_{\text{Euler}} = 100\times\frac{|0.5080-0.5082|}{0.5082} \simeq 0.04\%,\ \ \delta_{\text{Milstein}} = 100\times\frac{|0.5078-0.5082|}{0.5082} \simeq 0.08\%$$

$$\delta_{\text{Explicit}} = 100\times\frac{|0.5079-0.5082|}{0.5082} \simeq 0.06\%,\ \ \delta_{\text{WZ-AB}} = 100\times\frac{|0.5080-0.5082|}{0.5082} \simeq 0.04\%$$

$$\delta_{\text{WZ-PC}} = 100\times\frac{|0.5082-0.5082|}{0.5082} \simeq 0.00\%$$

It can be seen that other deterministic schemes such as Runge-Kutta can also be used with Wong-Zakai scheme and similar relative errors are obtained from the simulations. Wong-Zakai scheme can be seen to produce similar results and errors compared to the popular stochastic approximation methods.

Problem 2. (A solar irradiation problem) Solar radiation is one of the leading clean energy sources and modelling studies for efficient harvesting to produce clean power has been widely studied in the last decades [22]. There are various numerical weather prediction (NWP) models and simulation systems to predict the solar irradiation within other meteorological parameters to increase productivity. One of the methods for the prediction of solar radiation is the use of SDE. The SDE are indeed a useful tool for modelling the volatility in power production through changes in solar radiation levels due to the non-stable nature of atmospheric conditions such as dust, water vapor, air pollution and, *etc*. [22]. A SDE with additive noise has been given by Iversen *et al*. for solar irradiance prediction within a model that tracks the NWP [5]. The equation is given:

$$\mathrm{d}X_t = \theta_x \left(n_t \mu_x - X_t \right) \mathrm{d}t + \sigma_x \mathrm{d}W_t \tag{13}$$

where X_t is the actual solar irradiance at time t. The parameters are n_t respresents the predicted radiance at t, μ_x is a scaling parameter for n_t, θ_x determines the rate of model reversion predicted irradiance level and σ_x is the additive noise rate. The original model tracks the NWP provided by the Danish Meteorological Institute. Our application uses solar irradiance prediction data obtained for Rize (Turkey) city center (41.0284 N, 40.5157 E) from Copernicus Atmosphere Monitoring Service. The data has been generated using Copernicus Atmosphere Monitoring Service Information 2018, [23].

Equation (13) is an Ito SDE with additive noise and hence, its Stratonovich SDE version is obtained:

$$\mathrm{d}X_t = \theta_x \left(n_t \mu_x - X_t \right) \mathrm{d}t + \sigma_x \circ \mathrm{d}W_t \tag{14}$$

Using Wong-Zakai approximation with Adams Bashforth method and predictor-corrector method with Adams-Moulton as the corrector pair for eq. (14), we obtain the following results, tab. 2. The time variable t is measured in hours for this problem and a time interval of [0, 48] has been investigated modelling two days of solar irradiance levels in Rize city center on August 15-16, 2018. A discretized sub-interval length $\Delta_n = 1$ was used for the stochastic methods and $h = 0.1$ was used for the deterministic methods used with Wong-Zakai method within each hourly interval, tab. 2. An extra simulation of Euler method with $\Delta_n = 0.01$ has also been added for comparison. All of the methods have been simulated for 10^5 times. The solution curves have been shown in the fig. 3. The initial condition for the analysis has been used as $X_0 = 0$. Global irradiation on horizontal plane data has been referred in the simulations. Results for the second day are also given in the tab. 3.

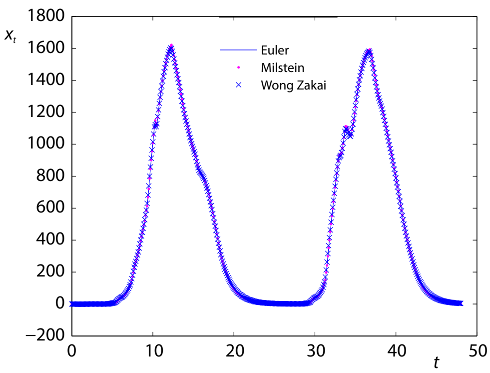

Figure 3. Approximate solutions of eq. (14)

Since there is no exact solution for the SDE the errors cannot be accurately compared. However, if the most basic and widely used method – Euler method (with $\Delta_n = 0.01$) – is used for comparison, we see the relative error percentages:

$$\delta_{\text{Euler}} = 100 \times \frac{|0.0057 - 0.0062|}{0.0062} \simeq 8.06\%, \quad \delta_{\text{Milstein}} = 100 \times \frac{|0.0050 - 0.0062|}{0.0062} \simeq 19.35\%$$

$$\delta_{\text{RK4}} = 100 \times \frac{|0.0064 - 0.0062|}{0.0062} \simeq 3.23\%, \quad \delta_{\text{Explicit}} = 100 \times \frac{|0.0055 - 0.0062|}{0.0062} \simeq 11.29\%$$

$$\delta_{\text{WZ-AB}} = 100 \times \frac{|0.0064 - 0.0062|}{0.0062} \simeq 3.23\%, \quad \delta_{\text{WZ-PC}} = 100 \times \frac{|0.0062 - 0.0062|}{0.0062} \simeq 0.00\%$$

Table 2. Comparison of results for eq. (14) using 10^5 simulations in $t \in [0, 24]$ ($\times 10^3$ Wh/m^2)

t	Expected value, $E(X_t)$						
	Stochastic schemes					Wong-Zakai Scheme with	
	Euler	Euler, $\Delta = 0.01$	Milstein	Explicit	RK4	Adams-B	Predictor-C
0	0	0	0	0	0	0	0
1	0.0002	0.0002	0.0001	0.0000	0.0002	0.0001	0.0002
2	0.0000	0.0000	0.0001	0.0000	0.0004	0.0000	0.0000
3	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
4	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0052	0.0049	0.0045	0.0049	0.0054	0.0049	0.0049
6	0.0383	0.0380	0.0382	0.0385	0.0387	0.0383	0.0380
7	0.0935	0.0935	0.0932	0.0932	0.0963	0.0974	0.0935
8	0.2812	0.2785	0.2808	0.2808	0.2826	0.2863	0.2785
9	0.5211	0.5195	0.5206	0.5203	0.5266	0.5285	0.5195
10	1.0074	1.0052	1.0072	1.0063	1.0164	1.0154	1.0052
11	1.3289	1.3045	1.3284	1.3284	1.3051	1.3302	1.3045
12	1.5871	1.5690	1.5871	1.5868	1.5678	1.5788	1.5690
13	1.4501	1.4346	1.4499	1.4497	1.4253	1.4342	1.4346
14	1.1721	1.1680	1.1724	1.1723	1.1609	1.1673	1.1680
15	0.9707	0.9725	0.9713	0.9710	0.9685	0.9707	0.9725
16	0.8100	0.8139	0.8100	0.8095	0.8123	0.8138	0.8139
17	0.6671	0.6674	0.6672	0.6666	0.6622	0.6626	0.6674
18	0.4101	0.4120	0.4105	0.4099	0.4063	0.4068	0.4120
19	0.1998	0.2047	0.2004	0.1993	0.2028	0.2035	0.2047
20	0.0968	0.1019	0.0970	0.0964	0.1013	0.1011	0.1019
21	0.0468	0.0504	0.0470	0.0470	0.0502	0.0498	0.0504
22	0.0227	0.0251	0.0229	0.0231	0.0250	0.0247	0.0251
23	0.0111	0.0126	0.0107	0.0114	0.0133	0.0128	0.0126
24	0.0057	0.0062	0.0050	0.0055	0.0064	0.0064	0.0062

As it was aforementioned, the relative error percentages (percent errors) are found as:

$$\delta_{\text{Euler}} = 100 \times \frac{|0.0035 - 0.0038|}{0.0038} \simeq 7.89\%, \quad \delta_{\text{Milstein}} = 100 \times \frac{|0.0034 - 0.0038|}{0.0038} \simeq 10.53\%$$

$$\delta_{\text{Explicit}} = 100 \times \frac{|0.0037 - 0.0038|}{0.0038} \simeq 2.63\%, \quad \delta_{\text{WZ-AB}} = 100 \times \frac{|0.0039 - 0.0038|}{0.0038} \simeq 2.63\%$$

$$\delta_{\text{WZ-PC}} = 100 \times \frac{|0.0034 - 0.0038|}{0.0038} \simeq 7.89\%$$

Table 3. Comparison of results for eq. (14) using 10^5 simulations in $t \in [24, 48]$ ($\times 10^3$ Wh/m²)

t	Expected value, $E(X_t)$						
	Stochastic schemes					Wong-Zakai Scheme with	
	Euler	Euler (10^6)	Milstein	Explicit	RK4	Adams-B	Predictor-C
24	0.0057	0.0062	0.0050	0.0055	0.0064	0.0064	0.0064
25	0.0029	0.0029	0.0024	0.0027	0.0031	0.0028	0.0032
26	0.0011	0.0012	0.0015	0.0018	0.0018	0.0013	0.0020
27	0.0006	0.0005	0.0010	0.0008	0.0011	0.0004	0.0010
28	0.0004	0.0005	0.0004	0.0001	0.0008	0.0001	0.0004
29	0.0038	0.0046	0.0050	0.0050	0.0051	0.0047	0.0045
30	0.0399	0.0398	0.0405	0.0409	0.0402	0.0414	0.0401
31	0.0977	0.0985	0.0981	0.0982	0.1028	0.1008	0.1050
32	0.4995	0.5024	0.4996	0.5001	0.5143	0.5095	0.5129
33	0.9300	0.9237	0.9300	0.9304	0.9271	0.9286	0.9165
34	1.1079	1.0929	1.1077	1.1083	1.0889	1.0999	1.0873
35	1.2334	1.2326	1.2335	1.2335	1.2412	1.2502	1.2414
36	1.5153	1.5085	1.5151	1.5151	1.5110	1.5154	1.5110
37	1.5524	1.5375	1.5514	1.5519	1.5278	1.5377	1.5312
38	1.2761	1.2736	1.2763	1.2766	1.2684	1.2739	1.2712
39	1.0522	1.0506	1.0524	1.0523	1.0439	1.0484	1.0467
40	0.7924	0.7945	0.7926	0.7927	0.7874	0.7915	0.7880
41	0.5007	0.5046	0.5013	0.5009	0.4992	0.5023	0.4998
42	0.2810	0.2863	0.2814	0.2816	0.2833	0.2850	0.2827
43	0.1367	0.1419	0.1364	0.1364	0.1411	0.1420	0.1406
44	0.0661	0.0704	0.0662	0.0663	0.0702	0.0707	0.0698
45	0.0321	0.0347	0.0324	0.0318	0.0348	0.0354	0.0344
46	0.0153	0.0170	0.0157	0.0150	0.0176	0.0176	0.0173
47	0.0073	0.0082	0.0077	0.0067	0.0085	0.0085	0.0087
48	0.0035	0.0038	0.0034	0.0037	0.0038	0.0039	0.0041

It is seen that the Wong-Zakai method with the Adams-Bashforth and Adams-Bashorth-Moulton predictor-corrector pairs gives the results with least error at t = 24. Note that the values of the parameters for eq. (14) have been used for the numerical simulations:

$$\theta_x = 0.699, \ \ \mu_x = 0.845$$

and the noise scaling parameter has been used as σ_x = 113 [5].

Problem 3. (An electric circuit problem): A RC circuit with constant parameters has been given as [4]:

$$R\frac{\mathrm{d}Q(t)}{\mathrm{d}t}+\frac{1}{C}Q(t)=V(t)+\alpha(t)W(t), \ \ Q(0)=Q_0 \tag{15}$$

where $Q(t)$ is the electric charge at time t, $V(t)$ – the voltage at t and $\alpha(t)$ show the noise intensity with $W(t) = (\mathrm{d}/\mathrm{d}t)B(t)$ with the Brownian motion $B(t)$.

If

$$V(t) = e^t,\ \alpha(t) = \frac{1}{25}\sin(t),\ R = 1,\ C = 2$$

case with $Q_0 = 3$ is considered, we get Ito SDE:

$$\mathrm{d}Q_t = \left(e^t - \frac{1}{2}Q_t\right)\mathrm{d}t + \frac{\sin(t)}{25}dW(t),\ Q(0) = 3 \tag{16}$$

The expected value for $Q(t)$ is given:

$$E[Q(t)] = e^{-t/RC}\left[3 + \frac{1}{R}\int_0^t e^{s/RC}V(s)\mathrm{d}s\right] \tag{17}$$

in [4] and for $V(t) = e^t$, $\alpha(t) = 1/25\sin(t)$, $R = 1$, $C = 2$, we get:

$$E[Q(t)] = \frac{7}{3}e^{-t/2} + \frac{2}{3}e^t \tag{18}$$

For the Ito SDE (16), we obtain the following Stratonovich SDE:

$$\mathrm{d}Q_t = \left(e^t - \frac{1}{2}Q_t\right)\mathrm{d}t + \frac{\sin(t)}{25}\circ\mathrm{d}W(t) \tag{19}$$

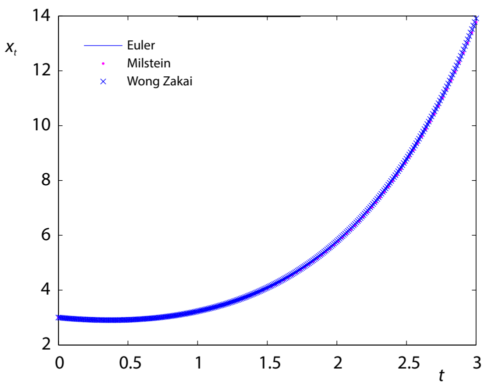

Figure 4. Approximate solutions of eq. (19)

since the SDE contains additive noise. Comparison of the expected value eq. (18) with the approximate expected values obtained from the previously mentioned methods is given in the following tab. 4. The solutions through Euler, Milstein, and Wong-Zakai schemes are shown in fig. 4.

The numerical results in tab. 4 have been obtained for a sub-interval length of $\Delta_n = 0.1$ for the stochastic methods and $h = 0.1$ for the deterministic methods used with Wong-Zakai method. The 10^5 simulations have been used for approximate expectations. The relative error percentages, relative to the expected value eq. (18), are found:

$$\delta_{\text{Euler}} = 100\times\frac{|3.2210 - 3.2274|}{3.2274} \simeq 0.198\%,\ \delta_{\text{Milstein}} = 100\times\frac{|3.2208 - 3.2274|}{3.2274} \simeq 0.204\%$$

$$\delta_{\text{WZ-AB}} = 100\times\frac{|3.2275 - 3.2274|}{3.2274} \simeq 0.003\%,\ \delta_{\text{WZ-PC}} = 100\times\frac{|3.2275 - 3.2274|}{3.2274} \simeq 0.003\%$$

Table 4. Comparison of results for eq. (19) using 10^5 simulations

t	Expected value, $E(Q_t)$					(19)
	Stochastic schemes			Wong-Zakai with:		
	Euler	Milstein	RKIV	Adams-B.	Predictor-C.	
0.0	3	3	3	3	3	3
0.1	2.9557	2.9557	2.9563	2.9563	2.9563	2.9563
0.2	2.9243	2.9243	2.9256	2.9255	2.9256	2.9256
0.3	2.9064	2.9064	2.9082	2.9082	2.9082	2.9082
0.4	2.9025	2.9025	2.9049	2.9049	2.9049	2.9049
0.5	2.9133	2.9133	2.9164	2.9163	2.9163	2.9163
0.6	2.9396	2.9396	2.9433	2.9433	2.9433	2.9433
0.7	2.9824	2.9824	2.9868	2.9868	2.9868	2.9868
0.8	3.0428	3.0427	3.0477	3.0478	3.0478	3.0478
0.9	3.1218	3.1218	3.1275	3.1275	3.1276	3.1275
1.0	3.2210	3.2208	3.2274	3.2275	3.2275	3.2274

Table 5. Results for eq. (19) with varying number N of sub-intervals ($\Delta_n = T/N$ for $T = 1$)

t	Euler-Maruyama				Milstein			$E(Q_t)$
	100	200	400	1000	100	200	400	
0.0	3	3	3	3	3	3	3	3
0.1	2.9557	2.9560	2.9562	2.9563	2.9557	2.9560	2.9562	2.9563
0.2	2.9243	2.9249	2.9252	2.9254	2.9243	2.9249	2.9253	2.9256
0.3	2.9064	2.9073	2.9078	2.9080	2.9064	2.9073	2.9078	2.9082
0.4	2.9025	2.9037	2.9043	2.9047	2.9025	2.9037	2.9043	2.9049
0.5	2.9133	2.9148	2.9156	2.9160	2.9133	2.9148	2.9156	2.9163
0.6	2.9396	2.9414	2.9424	2.9429	2.9396	2.9415	2.9424	2.9433
0.7	2.9824	2.9846	2.9857	2.9863	2.9824	2.9846	2.9857	2.9868
0.8	3.0428	3.0453	3.0465	3.0473	3.0427	3.0452	3.0466	3.0478
0.9	3.1218	3.1247	3.1261	3.1270	3.1218	3.1246	3.1261	3.1275
1.0	3.2210	3.2242	3.2258	3.2267	3.2208	3.2242	3.2258	3.2274

The difference between Wong-Zakai schemes with Adams-Bashforth and predictor-corrector methods and the other stochastic methods are abundantly clear for the results in tab. 4. Increase in the evaluation points for Milstein and Euler methods seem to decrease the error in the results, as seen in tab. 5. The N = 100 points have been used for the approximate expected value with Wong-Zakai schemes and almost 0.003% relative error has been obtained, whereas Euler and Milstein schemes have fail to produce results with such a small amount of error with more evaluation points.The N = 1000 points for Euler-Maruyama scheme results in a relative error of 0.022%, whereas this amount is 0.050% for N = 400 points and 0.099% for N = 200 points with both methods. Note that although stochastic Runge-Kutta method seems to have no error at t = 1.0, results show that there is a similar amount of errors for this method too. Increasing the simulation repetitions from 10^5-10^6 does little effect for the relative errors in Euler, Milstein and Runge-Kutta methods too. Milstein method with the same number of sub-intervals and 10^6 simulations gives 0.201% relative errors, which is not much different from 0.204% for 10 simulations.

Conclusion

In this study, Wong-Zakai approximation for Stratonovich SDE has been used with two deterministic approximation schemes, Adams-Bashforth and predictor-corrector method, to obtain approximate solutions for SDE appearing in modelling problems. The SDE examples from energy and electrical engineering problems were used to present the applications of the method. Examples show that Wong-Zakai convergence and deterministic approximation methods of differential equations provide a powerful approximation technique for SDE. This method is an effective and essential alternative for the investigation of SDE. Apart from providing an alternative to the popular methods like Euler-Maruyama, Milstein and Runge-Kutta, the method also provides the option to use various deterministic approximation schemes together with Wong-Zakai convergence. This flexible structure of the method provides an advantage in applications since it gives the option to choose the better deterministic scheme through the investigation of the structure of the SDE under consideration. Deterministic schemes like Adams-Bashforth, Heun, Euler and, *etc*. can be evaluated to obtain the best approximation with Wong-Zakai method. The examples also show that Wong-Zakai method can achieve a better convergence rate in some cases. In *Problem 3*, it is seen that Wong-Zakai approximation with deterministic Adams-Bashforth and predictor-corrector schemes achieves similar relative errors by using less than 10 times evaluation points compared to the stochastic Euler scheme, which is one of the most popular stochastic approximation methods. The 10 times more evaluation points means much more computation time and burden, even for computer assisted calculations. Hence, the Wong-Zakai alternative promises a significantly smaller amount of workload and time in some applications. Similar simulation numbers were used for all of the methods in applications to obtain the approximate expectations and Wong-Zakai method was found to be performing at least on the same level as Euler-Maruyama and Milstein methods. Considering these two methods are widely used in applications appearing in engineering, finance, biology and, *etc*., this method should be considered for the investigation of SDE since it promises better accuracy and approximations whenever Wong-Zakai convergence can be used. Note that the deterministic schemes with better convergence rates could be used with Wong-Zakai approximation obtain even better results against Euler and Milstein schemes. A thorough investigation was performed with various stochastic methods, MATLAB-SDE algorithm, Monte-Carlo simulations and the expectations of the SDE to present the results. All of the stochastic investigation methods show that Wong-Zakai is an accurate and reliable alternative tool for SDE.

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References

[1] Merdan, M., *et al*., Comparison of Stochastic and Random Models for Bacterial Resistance, *Advances in Difference Equations*, *2017* (2017), 1, 133
[2] Salgia, R., *et al*., Modelling Small Cell Lung Cancer (SCLC) Biology through Deterministic and Stochastic Mathematical Models, *Oncotarget*, *9* (2018), 40, 26226
[3] Moller, J. K., *et al*., Parameter Estimation in a Simple Stochastic Differential Equation for Phytoplankton Modelling, *Ecological Modelling*, *222* (2011), 11, pp. 1793-1799

[4] Khodabin, M., Rostami, M., Mean Square Numerical Solution of Stochastic Differential Equations by Fourth Order Runge-Kutta Method and Its Application in the Electric Circuits with Noise, *Advances in Difference Equations*, *2015* (2015), 1, 62
[5] Iversen, E. B., *et al*., Probabilistic Forecasts of Solar Irradiance Using Stochastic Differential Equations, *Environmetrics*, *25* (2014), 3, pp. 152-164
[6] Wang, J., Ahmed, H. M., Null Controllability of Non-Local Hilfer Fractional Stochastic Differential Equations, *Miskolc Mathematical Notes*, *18* (2017), 2, pp. 1073-1083
[7] Kloeden, P. E., Platen, E., *Numerical Solution of Stochastic Differential Equations*, Springer Science & Business Media, New York, USA, 2013
[8] Gard, T. C., *Introduction to Stochastic Differential Equations*, Monographs and Text-Books in Pure and Applied Mathematics, Marcel Dekker Inc., New York, USA, 1988
[9] El-Tawil, M., *et al*., A Proposed Technique of SFEM on Solving Ordinary Random Differential Equation, *Applied Mathematics and Computation*, *161* (2005), 1, pp. 35-47
[10] Casaban, M. C., *et al.*, Determining the First Probability Density Function of Linear Random Initial Value Problems by the Random Variable Transformation (RVT) Technique: A Comprehensive Study, *Abstract and Applied Analysis*, *2014* (2014), ID248512
[11] Wong, E., Zakai, M., On the Convergence of Ordinary Integrals to Stochastic Integrals, *The Annals of Mathematical Statistics*, *36* (1965), 5, pp. 1560-1564
[12] Wong, E., Zakai, M., On the Relation between Ordinary and Stochastic Differential Equations, *International Journal of Engineering Science*, *3* (1965), 2, pp. 213-229
[13] Wang, X., *et al.*, Wong-Zakai Approximations and Attractors for Stochastic Reaction-Diffusion Equations on Unbounded Domains, *Journal of Differential Equations*, *264* (2018), 1, pp. 378-424
[14] Flandoli, F., *et al*., Well-Posedness of the Transport Equation by Stochastic Perturbation, *Inventiones Mathematicae*, *180* (2010), 1, pp.1-53
[15] Londono, J. A., *et al*., Numerical Performance of some Wong-Zakai Type Approximations for Stochastic Differential Equations, *International Journal of Pure and Applied Mathematics*, *107* (2016), 2, pp. 301-315
[16] Kelly, D., *et al*., Smooth Approximation of Stochastic Differential Equations, *The Annals of Probability*, *44* (2016), 1, pp. 479-520
[17] Zhang, T., Strong Convergence of Wong-Zakai Approximations of Reflected SDEs in a Multidimensional General Domain, *Potential Analysis*, *41* (2014), 3, pp. 783-815
[18] Nakao, S., On Weak Convergence of Sequences of Continuous Local Martingales, *Annales de l'IHP Probabilités et Statistiques*, *22* (1986), 3, pp. 371-380
[19] Hairer, M., Pardoux, E., A Wong-Zakai Theorem for Stochastic PDEs, *Journal of the Mathematical Society of Japan*, *4* (2015), 67, pp. 1551-1604
[20] Quarteroni, A., *et al*., *Numerical Mathematics*, Springer Science & Business Media, New York, USA, 2010
[21] Cyganowski, S., , *et al*., *From Elementary Probability to Stochastic Differential Equations with MAPLE*, Springer Science & Business Media, New York, USA, 2001
[22] Voyant, C., *et al*., Numerical Weather Prediction (NWP) and Hybrid ARMA/ANN Model to Predict Global Radiation, *Energy*, *39* (2012), 1, pp. 341-355
[23] ***, Solar Irradiance Data for 41.02845 N, 50.5157E August 15, 2018, August 21, 2018, http://www.soda-pro.com/

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