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Aggregation operators of complex fuzzy Z-number sets and their applications in multi-criteria decision making

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Abstract
Fuzzy sets (FSs) are a flexible and powerful tool for reasoning about uncertain situations that cannot be adequately expressed by classical sets. However, these sets fall short in two areas. The first is the reliability of this tool. Z-numbers are an extension of fuzzy numbers that improve the representation of uncertainty by combining two important components: restriction and reliability. The second is the problems that need to be solved simultaneously. Complex fuzzy sets (CFSs) overcome this problem by adding a second dimension to fuzzy numbers and simultaneously adding connected elements to the solution. However, they are insufficient when it comes to problems involving these two areas. We cannot express real-life problems that need to be solved at the same time and require the reliability of the information given with any set approach given in the literature. Therefore, in this study, we propose the complex fuzzy Z-number set (CFZNS), a generalization of Z-numbers and CFS, which fills this gap. We provide the operational laws of CFZNS along with some properties. Additionally, we define two essential aggregation operators called complex fuzzy Z-number weighted averaging (CFZNWA) and complex fuzzy Z-number weighted geometric (CFZNWG) operators. Then, we present an illustrative example to demonstrate the proficiency and superiority of the proposed approach. Thus, we process multiple fuzzy expressions simultaneously and take into account the reliability of these fuzzy expressions in applications. Furthermore, we compare the results with the existing set operations to confirm the advantages and demonstrate the efficiency of the proposed approach. Considering the simultaneous expression of fuzzy statements, this study can serve as a foundation for new aggregation operators and decision-making problems and can be extended to many new applications such as pattern recognition and clustering.

Keywords Complex fuzzy Z-numbers (CFZN) · Z-number operations · Complex fuzzy Z-number weighted averaging (CFZNWA) operator · Complex fuzzy Z-number weighted geometric (CFZNWG) operator

Abbreviations

FS	Fuzzy set
CFS	Complex fuzzy set
CFZNS	Complex fuzzy Z-number set
CFZNWA	Complex fuzzy Z-number weighted averaging
CFZNWG	Complex fuzzy Z-number weighted geometric operators
μ_c	Complex membership function
T	T-norm
S	T-Conorm
Z_c	Complex fuzzy Z-number set
$\mathbb{C}_z$	Complex valued restriction
$\omega_{\mathbb{C}_z}$	Coefficient of complex angle
$\mathbb{r}_z$	Complex reliability value
$\omega_{\mathbb{r}}$	Coefficient of complex reliability angle
$S(z)$	Score function of complex fuzzy Z-numbers

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$CV(z)$	C-accuracy function of complex fuzzy Z-numbers
$RV(z)$	R-accuracy function of complex fuzzy Z-numbers
i	Set of complex Z-number elements ($i = 1, 2, \ldots, n$)
k	Same indices as i in proofs
z_i	Collection of CFZNs
α_i	Weight vector of z_i
MCDM	Multi-criteria decision-making
$A = \{A_1, A_2, \ldots, A_m\}$	Set of m alternatives
$C = \{C_1, C_2, \ldots, C_n\}$	Set of n criteria
D	Decision matrix
N_{ij}	Normalized decision matrix
R_i	Aggregated values of decision matrix.
UAV	Unmanned aerial vehicles
UCAV	Unmanned combat aerial vehicles

Introduction

Knowledge is one of the most significant phenomena in the history of the world. Acquiring and utilizing information plays a crucial role in technological advancements. While classical sets were sufficient for these developments until the 1950s, they proved inadequate in uncertain situations brought about by new technologies. To address such uncertainties, Zadeh [44] expanded classical sets and introduced fuzzy sets. This expansion involves allowing for degrees of membership of elements in sets, unlike classical sets where an element either belongs or does not belong. This enables a better representation of real-world problems with uncertainty using fuzzy sets. In other words, by incorporating the concept of partial membership into sets, fuzzy sets provide a flexible and powerful tool for reasoning about uncertain situations that cannot be adequately expressed by classical sets. Despite being studied for many years, both fuzzy sets and their extensions [10, 11, 34] continue to find applications in various fields, including artificial intelligence [14, 36], decision making [9, 13, 19, 21], control systems [29, 46], operations research [17, 23, 24], supply chain [25, 26], pattern recognition [18, 20, 40], and others.

Introduced by the pioneer of fuzzy logic, Zadeh [45], Z-numbers are an extension of fuzzy numbers that enhance the representation of uncertainty by incorporating two crucial components: constraint and reliability. Z-numbers provide a comprehensive framework for modelling and managing uncertainty, considering both the degree to which a value belongs to a set and the degree of confidence associated with that value. In traditional fuzzy set theory, fuzzy numbers are used to represent imprecise or uncertain quantities by assigning a membership degree to each value. However, fuzzy numbers primarily focus on capturing the degree of membership without explicitly considering the reliability of membership. On the other hand, Z-numbers extend the concept of fuzzy numbers by incorporating the concept of reliability, which reflects the level of certainty or confidence associated with the degree of membership. The degree of constraint in Z-numbers represents the extent to which a value belongs to a set, similar to the degree of membership in fuzzy numbers. Conversely, the degree of reliability in Z-numbers indicates the level of confidence or reliability associated with the degree of constraint. It reflects the quality of the data used to determine the reliability or degree of constraint of available information. The relationship between Z-numbers and fuzzy numbers can be seen as an extension or enrichment of fuzzy numbers. By combining both degrees of constraint and reliability, Z-numbers offer a more nuanced and comprehensive representation of uncertainty. This additional information allows for more accurate modelling and reasoning in uncertain or imprecise domains, enabling better decision-making processes in complex real-world situations. Similar to fuzzy sets, Z-numbers have also gained significant attention in the literature [4, 7, 8, 35, 43].

Complex fuzzy sets (CFSs) are an advanced extension of fuzzy sets that allow for a more flexible representation of uncertainty by characterizing membership functions in the complex plane and introduced by Ramot et al. [31]. CFSs can represent partial information deficiency of data and its fluctuations at a certain time stage during execution. Unlike traditional fuzzy sets, which assign membership values in the range of [0, 1], complex fuzzy sets extend the range of membership values up to the unit circle in the complex plane. In complex fuzzy sets, the membership function assigns each element a complex number that represents both the degree of membership and the phase angle associated with it. The magnitude of the complex number represents the degree of membership, while the phase angle captures the uncertainty or ambiguity in a particular direction. By utilizing the complex plane, complex fuzzy sets can effectively model situations where uncertainties exhibit directionality or when there are correlations among different membership degrees. This allows for a richer and more accurate representation of complex and uncertain phenomena, enabling improved reasoning and decision-making processes. Their ability to capture both the degree of membership and the uncertainty aspect provides a powerful tool for modelling and analysing complex systems with imprecise or uncertain information. Complex fuzzy sets find applications in various fields, including image processing [6], decision making [28] and pattern recognition [27].

There are cases where Z-numbers and complex fuzzy sets cannot express and fall short when it comes to decision making. We know that CFSs can represent the partial ignorance of problems and fluctuations at a certain stage of time during execution, adding a second dimension to fuzzy numbers. We also know that Z-numbers add to the reliability of the given fuzzy values. However, we cannot measure the reliability of complex fuzzy numbers, nor can we use Z-numbers in multidimensional problems that require simultaneous solutions. Additionally, we can't measure the reliability part of the Z-numbers. Therefore, the reliability part of the Z-numbers has to be considered as 1 when expressing the reliability and performing operations on it and it causes the loss of information. Especially in companies that trade in the software sector, there are points where the known set approaches are insufficient. For example, suppose a business firm purchases software from a software company. There are four main parameters here. The software version and updates to be received are two of them. However, as new versions come out, the removal of the restriction of the old version or the reliability of its removal is another parameter. In addition, how long the updates will be given is another parameter. And these four parameters need to be processed simultaneously. Neither CFSs nor Z-numbers can handle such problems since the problem cannot be expressed by these two approaches. Motivated by these shortcomings, in this paper, we propose a new approach that is called Complex fuzzy Z-number set (CFZNS) that can express and solve these problems. The CFZN set to be defined consists of two elements such as Z-numbers, complex restriction, and complex reliability, respectively. Each element has also two components, complex value, and coefficient of complex angle for complex restriction, complex reliability, and the coefficient reliability angle. Therefore, the CFZN set to be defined is a set that can measure both the constraints and the reliability of the constraints in the multidimensional problems. The features of the CFZNS and the contributions of the article to the literature can be summarized as follows:

- CFZNS is a new approach to solve the above-mentioned shortcomings.
- CFZNS is a generalization of both Z-numbers and CFSs.
- Both the operation laws of the CFZNS are investigated and the score and accuracy functions are given to measure the relationship of the numbers in this set with each other.
- Two most important and fundamental aggregation operators are defined, and their desirable properties are discussed.
- To demonstrate the impact and scope of the proposed set, a numerical example is given and discussed, which cannot be expressed with the existing approaches in the literature.
- While the proposed method is not specifically designed for classification tasks, it demonstrates a high classification rate when applied to a real-world dataset (Iris flower data) from the UCI Machine Learning Repository.

The rest of the article is organized as follows: In Sect. "Preliminaries", we provide some preliminary information about fuzzy sets, Z-numbers, complex fuzzy sets, and t-norms. In Sect. "Complex Fuzzy Z-number Set", we introduce a novel approach called complex fuzzy Z-number set and present its properties. Then, we define two basic aggregation operators with useful properties. In Sect. "Numerical Application Using CFZNWA and CFZNWG Operators", we present a numerical example that utilizes the proposed aggregation operators under the proposed CFZNS, along with a validation test. Then, we compare the proposed approach with reduced sets and discuss its advantages. In Sect. "Numerical Application with a Real Dataset", we evaluate the effectiveness of CFZNs by applying them to a real-world dataset and then we compare the results with existing methods to assess the performance of CFZNs. Finally, in Sect. "Conclusion", we summarize the study and present our conclusions.

Preliminaries

In this section, some fundamental concepts related to complex fuzzy sets, Z-numbers and their properties are provided.

Definition 1 (Zadeh 1965) A fuzzy set A is defined on a universe of discourse X as:

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\} \tag{1}$$

where $\mu_A : X \rightarrow [0, 1]$ is the membership function of the fuzzy set A and $\mu_A(x)$ is the membership of $x \in X$ in A.

Definition 2 (Zadeh 2011) A Z-number is represented as an ordered pair of fuzzy numbers denoted by

$$Z = (A, B) \tag{2}$$

that is associated with a real valued uncertain variable X. Here, A is a restriction (constraint) on the values which X is allowed to take and B is a measure of reliability (certainty) of the first component.

Definition 3 [31] A complex fuzzy set C is defined on a universe of discourse X as:

$$C = \{\langle x, \mu_c(x)\rangle : x \in X\} \tag{3}$$

where μ_c is a membership function that assign any $x \in X$ as a complex valued grade of membership. Here, $\mu_c(x) = r_c(x)e^{i2\pi\omega_c(x)}$ that lies in a unit circle in the complex plane where $i = \sqrt{-1}$, $\mu_c(x) \in [0, 1]$ and $\omega_c(x)$ is real valued.

Definition 4 [16] A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t-norm if it satisfies the following conditions:

(1) $T(1, x) = x,$ for all x.
(2) $T(x, y) = T(y, x),$ for all x and y.
(3) $T(x, T(y, z)) = T(T(x, y), z),$ for all x, y, z.
(4) If $x \leq x'$ and $y \leq y',$ then $T(x, y) \leq T(x', y')$. $\qquad (4)$

Definition 5 [16] A function $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t-conorm if it satisfies the following conditions:

(1) $S(1, x) = x,$ for all x.
(2) $S(x, y) = S(y, x),$ for all x and y.
(3) $S(x, S(y, z)) = S(S(x, y), z),$ for all x, y, z.
(4) If $x \leq x'$ and $y \leq y',$ then $S(x, y) \leq S(x', y')$. $\qquad (5)$

The introduction of fuzzy sets required performing algebraic operations among fuzzy elements. Executing these operations with classical algebra would make the definition of fuzzy sets lose its meaning. Therefore, t-norm and t-conorm functions were inspired to define fuzzy algebraic operations. A t-norm function $T(x, y)$ and a t-conorm function $S(x, y)$ are called Archimedean if they are continuous and satisfy the conditions $T(x, x) < x$ and $S(x, x) > x$ for all $x \in (0, 1)$. Moreover, Klement and Mesiar [15] defined the strictly decreasing Archimedean t-norm and t-conorm using the additive generators g and h respectively as:

$$T(x, y) = g^{-1}(g(x) + g(y))$$
$$S(x, y) = h^{-1}(h(x) + h(y))$$

where $h(t) = g(1 - t)$ and $g : [0, 1] \rightarrow [0, \infty]$. Depending on the choice of the function $g(t)$, many types of t-norms and t-conorms can be obtained. If $g(t) = -\log t$, then $h(t) = -\log(1 - t)$ and we can get Algebraic t-norm and t-conorm, which is the basis of fuzzy operations, as follows:

$$S_A(x, y) = x + y - xy, \quad T_A(x, y) = xy \qquad (6)$$

Let $g(t) = \log\left(\frac{2-t}{t}\right)$, then $h(t) = g(1 - t) = \log\left(\frac{2-(1-t)}{1-t}\right)$ and it follows that $g^{-1}(t) = \frac{2}{e^t+1}$ and that $h^{-1}(t) = 1 - \frac{2}{e^t+1}$. Using the additive generators above, we have Einstein t-norm and t-conorm as:

$$S_E(x, y) = \frac{x + y}{1 + xy}, \quad T_E(x, y) = \frac{xy}{1 + (1 - x)(1 - y)} \qquad (7)$$

Complex fuzzy Z-number set

The one of the main aims of this work is to introduce the Complex fuzzy Z-numbers and investigate some of their properties.

Definition 6 A complex fuzzy Z-number set Z_c is defined on a universe of discourse X as:

$$Z = \{\langle x, C_z(x), R_z(x)\rangle : x \in X\} \qquad (8)$$

where $C_z : X \rightarrow \{a : a \in \mathbb{C}, |a| \leq 1\}$ denotes the complex valued restriction (constraint) and $R_z : X \rightarrow \{a : a \in \mathbb{C}, |a| \leq 1\}$ denotes the complex valued measure of reliability (certainty), given by:

$$C_z(x) = \mathbb{c}_z(x)e^{2\pi i \omega_{\mathbb{c}_z}(x)}, \; R_z(x) = \mathbb{r}_z(x)e^{2\pi i \omega_{\mathbb{r}_z}(x)},$$

and with the conditions:

$$\mathbb{c}_z(x), \mathbb{r}_z(x) \in [0, 1] \text{ and} \omega_{\mathbb{c}_z}(x), \omega_{\mathbb{r}_z}(x) \in [0, 1]$$

Then, complex fuzzy Z-numbers can be denoted as pairs $z = \langle \mathbb{c}e^{2\pi i \omega_{\mathbb{c}}}, \mathbb{r}e^{2\pi i \omega_{\mathbb{r}}}\rangle$ where $\mathbb{c}, \mathbb{r} \in [0, 1]$ are called complex value and reliability values, respectively and $\omega_{\mathbb{c}}, \omega_{\mathbb{r}} \in [0, 1]$ are called coefficient of complex angle and coefficient of reliability angle, respectively. We use the notation $z = \langle (c, \omega_c), (r, \omega_r)\rangle$ for convenience in complex operations. Note that

- if $\omega_{\mathbb{c}_z}(x) = \omega_{\mathbb{r}_z}(x) = 0$, then $z = \langle \mathbb{c}e^0, \mathbb{r}e^0\rangle = \langle \mathbb{c}, \mathbb{r}\rangle$, and CFZNs reduce to Z-numbers, which demonstrates that CFZNs are a generalization of Z-numbers.
- if $R_z(x) = \varnothing$, then CFZNS reduce complex fuzzy sets, which shows that CFZS is a generalization of complex fuzzy sets.

Definition 7 Let $A = \{\langle x, C_A(x), R_A(x)\rangle : x \in X\}$ and $B = \{\langle x, C_B(x), B(x)\rangle : x \in X\}$ be two CFZNSs. Then, the basic operations between them defines as follows:

$A \subseteq B$ if $c_A(x) \leq c_B(x), r_A(x) \leq r_B(x)$ and $\omega_{c_A}(x) \leq \omega_{c_B}(x), \omega_{r_A}(x) \leq \omega_{r_B}(x)$.
$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
$$A^c = \left(1 - c_A, \omega_{c_A}\right), \left(1 - r_A, \omega_{r_A}\right). \qquad (9)$$

Complex fuzzy Z-number operations

We define the complex fuzzy Z-number operations based on Archimedean t-norm and t-conorm inspired by Beliakov et al. [1]. According to them, we can use any pair of dual t-norm and t-conorm to define operations.

Definition 8 Let $z_1 = \langle (c_1, \omega_{c_1}), (r_1, \omega_{r_1}) \rangle$ and $z_2 = \langle (c_2, \omega_{c_2}), (r_2, \omega_{r_2}) \rangle$ be two CFZNs and λ is a scalar with $\lambda > 0$. Then the operational laws of CFZNs are defined as follows:

(1) $z_1 \oplus z_2$

$$= \langle (S_A(c_1, c_2), S_A(\omega_{c_1}, \omega_{c_2})), (S_E(r_1, r_2), S_E(\omega_{r_1}, \omega_{r_2})) \rangle$$

$$= \left\langle \begin{array}{c} \left(h_A^{-1}(h_A(c_1) + h_A(c_2)), h_A^{-1}(h_A(\omega_{c_1}) + h_A(\omega_{c_2})) \right), \\ \left(h_E^{-1}(h_E(r_1) + h_E(r_2)), h_E^{-1}(h_E(\omega_{r_1}) + h_E(\omega_{r_2})) \right) \end{array} \right\rangle$$

(2) $z_1 \otimes z_2$

$$= \langle (T_A(c_1, c_2), T_A(\omega_{c_1}, \omega_{c_2})), (T_E(r_1, r_2), T_E(\omega_{r_1}, \omega_{r_2})) \rangle$$

$$= \left\langle \begin{array}{c} \left(g_A^{-1}(g_A(c_1) + g_A(c_2)), g_A^{-1}(g_A(\omega_{c_1}) + g_A(\omega_{c_2})) \right), \\ \left(g_E^{-1}(g_E(r_1) + g_E(r_2)), g_E^{-1}(g_E(\omega_{r_1}) + g_E(\omega_{r_2})) \right) \end{array} \right\rangle$$

(3) $\lambda z_1 = \left\langle \left(h_A^{-1}(\lambda h_A(c_1)), h_A^{-1}(\lambda h_A(\omega_{c_1})) \right), \right.$
$$\left. \left(h_E^{-1}(\lambda h_E(r_1)), h_E^{-1}(\lambda h_E(\omega_{r_1})) \right) \right\rangle$$

(4) $z_1^{\lambda} = \left\langle \left(g_A^{-1}(\lambda g_A(c_1)), g_A^{-1}(\lambda g_A(\omega_{c_1})) \right), \right.$
$$\left. \left(g_E^{-1}(\lambda g_E(r_1)), g_E^{-1}(\lambda g_E(\omega_{r_1})) \right) \right\rangle \tag{10}$$

If we choose $g_A(t) = -\log t$ and $g_E(t) = \log\frac{2-t}{t}$, then we have the operations in Eq. (11).

Definition 9 Let $z_1 = \langle (c_1, \omega_{c_1}), (r_1, \omega_{r_1}) \rangle$ and $z_2 = \langle (c_2, \omega_{c_2}), (r_2, \omega_{r_2}) \rangle$ be two CFZNs and λ is a scalar with $\lambda > 0$. Then the operational laws of CFZNs are defined as follows:

(1) $z_1 \oplus z_2 = \left\langle \left(1 - \prod_{i=1}^{2}(1 - c_i), 1 - \prod_{i=1}^{2}(1 - \omega_{c_i}) \right), \right.$
$$\left. \left(\frac{r_1 + r_2}{1 + r_1 \cdot r_2}, \frac{\omega_{r_1} + \omega_{r_2}}{1 + \omega_{r_1} \cdot \omega_{r_2}} \right) \right\rangle$$

(2) $z_1 \otimes z_2 = \left\langle \left(\prod_{i=1}^{2} c_i, \prod_{i=1}^{2} \omega_{c_i} \right), \left(\frac{r_1 \cdot r_2}{1 + (1 - r_1) + (1 - r_2)}, \right. \right.$
$$\left. \left. \frac{\omega_{r_1} \cdot \omega_{r_2}}{1 + (1 - \omega_{r_1}) + (1 - \omega_{r_2})} \right) \right\rangle$$

(3) $\lambda z_1 = \left\langle \left(1 - (1 - c_1)^{\lambda}, 1 - (1 - \omega_{c_1})^{\lambda} \right), \right.$
$$\left(\frac{(1 + \omega_{r_1})^{\lambda} - (1 - \omega_{r_1})^{\lambda}}{(1 + \omega_{r_1})^{\lambda} + (1 - \omega_{r_1})^{\lambda}}, \right.$$
$$\left. \left. \frac{(1 + \omega_{r_1})^{\lambda} - (1 - \omega_{r_1})^{\lambda}}{(1 + \omega_{r_1})^{\lambda} + (1 - \omega_{r_1})^{\lambda}} \right) \right\rangle$$

(4) $z_1^{\lambda} = \left\langle \left(r_1^{\lambda}, \omega_{c_1}^{\lambda} \right), \left(\frac{2 r_1^{\lambda}}{(2 - r_1)^{\lambda} + r_1^{\lambda}}, \frac{2 \omega_{r_1}^{\lambda}}{(2 - \omega_{r_1})^{\lambda} + \omega_{r_1}^{\lambda}} \right) \right\rangle \tag{11}$

Theorem 1 Let $z_1 = \langle (c_1, \omega_{c_1}), (r_1, \omega_{r_1}) \rangle$ and $z_2 = \langle (c_2, \omega_{c_2}), (r_2, \omega_{r_2}) \rangle$ be two CFZNs and $\lambda > 0$ be a real number. Then, z_1^c, $z_1 \oplus z_2$, $z_1 \otimes z_2$, λz_1 and z_1^{λ} are also CFZNs.

Proof Since item 1 is obvious, and others are easy to see, we prove only item 2. Also, recall that $0 \leq c_1, c_2 \leq 1, 0 \leq \omega_{c_1}, \omega_{c_2} \leq 1, 0 \leq r_1, r_2 \leq 1$ and $0 \leq \omega_{r_1}, \omega_{r_2} \leq 1$,

For item 2, first, we show the constraint part of the CFZN. Since $0 \leq 1 - c_1 \leq 1$ and $0 \leq 1 - c_2 \leq 1$, we have $0 \leq \prod_{i=1}^{2}(1 - c_i) \leq 1$ which implies $0 \leq 1 - \prod_{i=1}^{2}(1 - c_i) \leq 1$. Additionally, $0 \leq 1 - \omega_{c_1} \leq 1$ and $0 \leq 1 - \omega_{c_2} \leq 1$ and thus, we have $0 \leq \prod_{i=1}^{2}(1 - \omega_{c_i}) \leq 1$, which also implies $0 \leq 1 - \prod_{i=1}^{2}(1 - \omega_{c_i}) \leq 1$.

To see the reliability part, we have $0 \leq r_1 + r_2 \leq 2$ and $1 \leq 1 + r_1 r_2 \leq 2$. Then, it concludes that $0 \leq \frac{r_1 + r_2}{1 + r_1 \cdot r_2} \leq 1$. Since the operations are same for ω we have $0 \leq \frac{\omega_{r_1} + \omega_{r_2}}{1 + \omega_{r_1} \cdot \omega_{r_2}} \leq 1$. Therefore $z_1 \oplus z_2$ is also a CFZN.

Theorem 2 Let z_1, z_2 and z_3 be three CFZNs. Then, the following equations are hold:

(1) $z_1 \oplus z_2 = z_2 \oplus z_1$

(2) $z_1 \otimes z_2 = z_2 \otimes z_1$

(3) $(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$

(4) $(z_1 \otimes z_2) \otimes z_3 = z_1 \otimes (z_2 \otimes z_3)$ \tag{12}

Theorem 3 Let z_1 and z_2 be two CFZNs and λ, λ_1 and λ_2 be three positive real numbers. Then, following equations are hold:

(1) $\lambda(z_1 \oplus z_2) = \lambda z_1 \oplus \lambda z_2$

(2) $(z_1 \otimes z_2)^{\lambda} = z_1^{\lambda} \otimes z_2^{\lambda}$

(3) $(\lambda_1 + \lambda_2) z_1 = \lambda_1 z_1 \oplus \lambda_2 z_1$

(4) $z_1^{\lambda_1 + \lambda_2} = z_1^{\lambda_1} \otimes z_1^{\lambda_2}$ \tag{13}

To compare CFZNs we need a function to measure them. When looking at previous works related to fuzzy complex numbers, this function is defined as the sum of the complex value and the coefficient of the complex angle, which in our case are c and ω_c. The same situation applies to IFSs. However, considering the Z-numbers, we need extended operations to measure them. Since the second part is the reliability part of the first part in a CFZN, the score function

can be defined as an expanded form of complex fuzzy numbers. On the other hand, if the scores of two different CFZNs are equal, the comparison becomes more complicated. For two CFZNs with equal scores, if their reliability values are equal, the reasonable approach is that the one with the larger complex value is considered the larger number. Similarly, for two equal CFZNs, if the complex values are the same, the one with the larger reliability value is considered the larger number.

Definition 10 Let $z = \langle (c, \omega_c), (r, \omega_r) \rangle$ be a CFZN. Then, the score function of z is defined as $S(z) = \frac{1}{4}(c + \omega_c) \cdot (r + \omega_r)$. If the score of two CFZNs is equal, then.

- If the reliability values are same, the C-accuracy function is defined as $CV(z) = c$.
- If the complex values are same, the R-accuracy function defined as $RV(z) = r$.

where $S(z)$, $CV(z)$, $RV(z) \in [0, 1]$. To compare two CFZNs z_1 and z_2, the following algorithm can be given:

(1) If $S(z_1) > S(z_2)$, then $z_1 > z_2$.
(2) If $S(z_1) = S(z_2)$ and
(i) If $r_{z1} = r_{z2}$ and

$$\begin{array}{ll} \text{a. If } CV(z_1) > CV(z_2) \text{ then } z_1 > z_2 \\ \text{b. If } CV(z_1) = CV(z_2) \text{ then } z_1 = z_2 \end{array} \quad (14)$$

(ii) If $c_{z1} = c_{z2}$ and
(c) If $RV(z_1) > RV(z_2)$ then $z_1 > z_2$

(d) If $RV(z_1) = RV(z_2)$ then $z_1 = z_2$

Example 1 Let $z_1 = \langle (0.5, 0.3), (0.4, 0.8) \rangle$ and $z_2 = \langle (0.4, 0.3), (0.6, 0.8) \rangle$. Then, the score values are calculated as $s(z_1) = 0.5 \times 0.4 + 0.3 \times 0.8 = 0.44$ and $s(z_2) = 0.4 \times 0.6 + 0.3 \times 0.8 = 0.48$. We obtain $s(z_1) \leq s(z_2)$, even though the first part of z_1 is greater than the first part of z_2. This is because the reliability of z_1 is lower than the reliability of z_2. This is consistent with the nature of the Z-numbers defined by Zadeh [45].

Example 2 Let $z_1 = \langle (0.7, 0.3), (0.6, 0.4) \rangle$, $z_2 = \langle (0.7, 0.3), (0.7, 0.3) \rangle$ and $z_3 = \langle (0.5, 0.5), (0.6, 0.4) \rangle$. Then the score values are $S(z_1) = S(z_2) = S(z_3) = 0.25$. Then, since $r_{z1} = r_{z3}$, $CV(z_1) = 0.7$ and $CV(z_3) = 0.5$, we get $z_1 > z_3$. Additionally, since $c_{z1} = c_{z2}$, $RV(z_1) = 0.6$ and $RV(z_2) = 0.7$, we have $z_2 > z_1$. Therefore, we have $z_2 > z_1 > z_3$.

Complex fuzzy Z-number aggregation operators

Fuzzy sets in a fuzzy environment consist of multiple uncertain elements. Similar to classical sets, there is a need for certain operators to handle such a group of numbers. The most fundamental and important operators in classical sets are the arithmetic mean and geometric mean operators. The fuzzy equivalent of these operations in fuzzy environments is represented by aggregation operators. The existence of these operators is inevitable to utilize sets in fuzzy environments that involve multiple elements in applications. By using algebraic operations, these operators are generalized, making them suitable for applications, especially in decision matrices, by reducing the data of many elements to a single entity, thereby enabling the measurement of data in fuzzy environments. Therefore, to process CFZNs in decision matrices in applications, in this section we will define the two most basic aggregation operators and examine their properties. In order to avoid redundancy in the definitions presented, we will use the notation $\mathcal{Z}$ to represent the set of all conceivable CFZNs.

Definition 11 Let $z_i (i = 1, 2, \ldots, n)$ be a collection of CFZNs. A complex fuzzy Z-number weighted averaging (CFZNWA) operator is a mapping $CFZNWA : \mathcal{Z}^n \to \mathcal{Z}$ defined by

$$\begin{aligned} & CFZNWA(z_1, z_2, \ldots, z_n) \\ &= \oplus_{i=1}^{n} (\alpha_i z_i) \\ &= \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \ldots \oplus \alpha_n z_n \end{aligned} \quad (15)$$

where $\alpha_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the weight vector of z_i with $\alpha_i \in [0, 1]$ and $\sum_{i=1}^{n} \alpha_i = 1$. If the weight vector is taken as $\alpha_i = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)$, the CFZNWA operator reduces to complex fuzzy Z-number averaging (CFZNA) operator as

$$CFZNA(z_1, z_2, \ldots, z_n) = \oplus_{i=1}^{n} z_i.$$

Theorem 4 Let $z_i (i = 1, 2, \ldots, n)$ be a collection of CFZNs. Then, the CFZNWA operator is also a CFZN and can be expressed as

$$\begin{aligned} & CFZNWA(z_1, z_2, \ldots, z_n) \\ &= \left\langle \begin{array}{c} \left(1 - \prod_{i=1}^{n} (1 - c_i)^{\alpha_i}, 1 - \prod_{i=1}^{n} \left(1 - \omega_{c_i} \right)^{\alpha_i} \right), \\ \left(\frac{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} - \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} + \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}, \frac{\prod_{i=1}^{n} (1+\omega_{r_i})^{\alpha_i} - \prod_{i=1}^{n} (1-\omega_{r_i})^{\alpha_i}}{\prod_{i=1}^{n} (1+\omega_{r_i})^{\alpha_i} + \prod_{i=1}^{n} (1-\omega_{r_i})^{\alpha_i}} \right) \end{array} \right\rangle \end{aligned} \quad (16)$$

Proof From Theorem 1, since the results all operations on CFZNs are also CFZN, it is obvious that the CFZNWA operator is also a CFNZ. To prove Eq. (16), we use the principle of mathematical induction on n.

I. For $n = 1$, then it is obvious that the equation holds.

II. Assume that the Eq. (16) holds for $n = k > 2$:

$CFZNWA(z_1, z_2, \ldots, z_k)$

$$= \left\langle \begin{array}{c} \left(1 - \prod_{i=1}^{k} (1 - c_i)^{\alpha_i}, \; 1 - \prod_{i=1}^{k} \left(1 - \omega_{c_i}\right)^{\alpha_i}\right), \\ \left(\frac{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} - \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} + \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}, \; \frac{\prod_{i=1}^{n} \left(1+\omega_{r_i}\right)^{\alpha_i} - \prod_{i=1}^{n} \left(1-\omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^{n} \left(1+\omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^{n} \left(1-\omega_{r_i}\right)^{\alpha_i}}\right) \end{array} \right\rangle$$

III. For $n = k + 1$, we need to show that $CFZNWA(z_1, z_2, \ldots, z_k, z_{k+1})$ holds for Eq. (15):

$CFZNWA(z_1, z_2, \ldots, z_k, z_{k+1})$

$= CFZNWA(z_1, z_2, \ldots, z_k) \oplus \alpha_{k+1} z_{k+1}$

$$= \left\langle \begin{array}{c} \left(1 - \prod_{i=1}^{k} (1 - c_i)^{\alpha_i}, \; 1 - \prod_{i=1}^{k} \left(1 - \omega_{c_i}\right)^{\alpha_i}\right), \\ \left(\frac{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} - \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} + \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}, \; \frac{\prod_{i=1}^{n} \left(1+\omega_{r_i}\right)^{\alpha_i} - \prod_{i=1}^{n} \left(1-\omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^{n} \left(1+\omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^{n} \left(1-\omega_{r_i}\right)^{\alpha_i}}\right) \end{array} \right\rangle$$

$$\oplus \left\langle \left(1 - (1 - c_{k+1})^{\alpha_{k+1}}, \; 1 - (1 - \omega_{k+1})^{\alpha_{k+1}}\right), (r_{k+1}, \omega_{r\,k+1}) \right\rangle$$

For the first part (restriction), using Eq. (11) we have:

$CFZNWA(z_1, z_2, \ldots, z_k) \oplus \alpha_{k+1} z_{k+1}$

$$= \left(\begin{array}{c} 1 - \left(1 - \prod_{i=1}^{k} (1 - c_i)^{\alpha_i}\right)\left(1 - (1 - c_{k+1})^{\alpha_{k+1}}\right), \\ 1 - \left(1 - \prod_{i=1}^{k} \left(1 - \omega_{c_i}\right)^{\alpha_i}\right)\left(1 - (1 - \omega_{k+1})^{\alpha_{k+1}}\right) \end{array} \right)$$

If we expand this statement, we have:

$CFZNWA(z_1, z_2, \ldots,$

$$z_k) \oplus \alpha_{k+1} z_{k+1} = \left(\begin{array}{c} \left(1 - \prod_{i=1}^{k+1} (1 - c_i)^{\alpha_i}\right), \\ \left(1 - \prod_{i=1}^{k+1} \left(1 - \omega_{c_i}\right)^{\alpha_i}\right) \end{array} \right)$$

For the second part (reliability), let $x_1 = \prod_{i=1}^{k} (1 + r_i)^{\alpha_i}$, $y_1 = \prod_{i=1}^{k} (1 - r_i)^{\alpha_i}$, $x_2 = (1 + r_{k+1})^{\alpha_{k+1}}$ and $y_2 = (1 - r_{k+1})^{\alpha_{k+1}}$. Also, same for ω_{c_i} let $\omega x_1 = \prod_{i=1}^{k} \left(1 + \omega_{c_i}\right)^{\alpha_i}$, $\omega y_1 = \prod_{i=1}^{k} \left(1 - \omega_{c_i}\right)^{\alpha_i}$, $\omega x_2 = \left(1 + \omega_{c_{k+1}}\right)^{\alpha_{k+1}}$ and $\omega y_2 = \left(1 - \omega_{c_{k+1}}\right)^{\alpha_{k+1}}$. We can rewrite the followings as:

$CFZNWA(z_1, z_2, \ldots, z_k)$

$$= \left(\frac{x_1 - y_1}{x_1 + y_1}, \; \frac{\omega x_1 - \omega y_1}{\omega x_1 + \omega y_1}\right), \alpha_{k+1} z_{k+1}$$

$$= \left(\frac{x_2 - y_2}{x_2 + y_2}, \; \frac{\omega x_2 - \omega y_2}{\omega x_2 + \omega y_2}\right)$$

Using the operational law in Eq. (11), we get the following:

$CFZNWA(z_1, z_2, \ldots, z_k) \oplus \alpha_{k+1} z_{k+1}$

$$= \left(\frac{\frac{x_1 - y_1}{x_1 + y_1} + \frac{x_2 - y_2}{x_2 + y_2}}{1 + \frac{x_1 - y_1}{x_1 + y_1} \cdot \frac{x_2 - y_2}{x_2 + y_2}}, \; \frac{\frac{\omega x_1 - \omega y_1}{\omega x_1 + \omega y_1} + \frac{\omega x_2 - \omega y_2}{\omega x_2 + \omega y_2}}{1 + \frac{\omega x_1 - \omega y_1}{\omega x_1 + \omega y_1} \cdot \frac{\omega x_2 - \omega y_2}{\omega x_2 + \omega y_2}}\right)$$

$$= \left(\frac{x_1 x_2 - y_1 y_2}{x_1 x_2 + y_1 y_2}, \; \frac{\omega x_1 \omega x_2 - \omega y_1 \omega y_2}{\omega x_1 \omega x_2 + \omega y_1 \omega y_2}\right)$$

$$= \left(\begin{array}{c} \frac{\prod_{i=1}^{k+1} (1 + r_i)^{\alpha_i} - \prod_{i=1}^{k+1} (1 - r_i)^{\alpha_i}}{\prod_{i=1}^{k+1} (1 + r_i)^{\alpha_i} + \prod_{i=1}^{k+1} (1 - r_i)^{\alpha_i}}, \\ \times \frac{\prod_{i=1}^{k+1} \left(1 + \omega_{r_i}\right)^{\alpha_i} - \prod_{i=1}^{k+1} \left(1 - \omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^{k+1} \left(1 + \omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^{k+1} \left(1 - \omega_{r_i}\right)^{\alpha_i}} \end{array} \right)$$

Therefore, Eq. (16) hold for all n and it completes the proof.

Aggregation operators on fuzzy environment should satisfy some fundamental mathematical properties. The developed $CFZNWA$ operator also meets the same specifications. To demonstrate these features and avoid repetition, let $z_i(i = 1, 2, \ldots, n)$ be a collection of CFZNs and $\alpha_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the weight vector of z_i with $\alpha_i \in [0, 1]$ and $\sum_{i=1}^{n} \alpha_i = 1$.

Property 1 (Idempotency). If all z_i are equal, i.e., $z_i = z = \langle (c, \omega_c), (r, \omega_r) \rangle$ for all i, then.

$$CFZNWA(z_1, z_2, \ldots, z_n) = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \ldots \oplus \alpha_n z_n = z. \tag{17}$$

Proof Since $z_i = z$ for all $i = 1, 2, \ldots, n$, we have

$CFZNWA(z_1, z_2, \ldots, z_n)$

$$= \left\langle \begin{array}{c} \left(1 - \prod_{i=1}^{n} (1 - c_i)^{\alpha_i}, \; 1 - \prod_{i=1}^{n} \left(1 - \omega_{c_i}\right)^{\alpha_i}\right), \\ \left(\frac{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} - \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} + \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}, \; \frac{\prod_{i=1}^{n} \left(1+\omega_{r_i}\right)^{\alpha_i} - \prod_{i=1}^{n} \left(1-\omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^{n} \left(1+\omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^{n} \left(1-\omega_{r_i}\right)^{\alpha_i}}\right) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \left(1 - (1 - c_i)^{\sum_{i=1}^{n} \alpha_i}, \; 1 - (1 - w_{ci})^{\sum_{i=1}^{n} \alpha_i}\right), \\ \left(\frac{(1+r_i)^{\sum_{i=1}^{n} \alpha_i} - (1-r_i)^{\sum_{i=1}^{n} \alpha_i}}{(1+r_i)^{\sum_{i=1}^{n} \alpha_i} + (1-r_i)^{\sum_{i=1}^{n} \alpha_i}}, \; \frac{(1+\omega_{r_i})^{\sum_{i=1}^{n} \alpha_i} - (1-\omega_{r_i})^{\sum_{i=1}^{n} \alpha_i}}{(1+\omega_{r_i})^{\sum_{i=1}^{n} \alpha_i} + (1-\omega_{r_i})^{\sum_{i=1}^{n} \alpha_i}}\right) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} (c_i, w_{ci}), \\ \left(\frac{(1+r_i) - (1-r_i)}{(1+r_i) + (1-r_i)}, \; \frac{(1+\omega_{r_i}) - (1-\omega_{r_i})}{(1+\omega_{r_i}) + (1-\omega_{r_i})}\right) \end{array} \right\rangle = \langle (c, \omega_c), (r, \omega_r) \rangle = z$$

Property 2 (Monotonicity). Let $z_i = \left\langle (c_i, \omega_{c_i}), (r_i, \omega_{r_i}) \right\rangle$ and $z_i^* = \langle (c_i^*, \omega_{c_i}^*), (r_i^*, \omega_{r_i}^*) \rangle$ be two collection of CFZNs where $i = 1, 2, \ldots n$. $z_i \leq z_i^*$ for all i, i.e., $c_i \leq c_i^*, r_i \leq r_i^*$, $\omega_{c_i} \leq \omega_{c_i}^*$ and $r_i \leq \omega_{r_i}^*$, then.

$$CFZNWA(z_1, z_2, \ldots, z_n) \leq CFZNWA(z_1^*, z_2^*, \ldots, z_n^*). \tag{18}$$

Note that since the second part is reliability, it should be $r_i \leq r_i^*$ and $\omega_{r_i} \leq \omega_{r_i}^*$. Otherwise, there would be no difference from IFNs.

Proof. We prove this property in two parts, restriction and reliability, to avoid confusion. For restriction part of CFZNWA operator, we have $c_i \leq c_i^*$ and $\omega_{c_i} \leq \omega_{c_i}^*$ for all i. We can write them as $1 - c_i \geq 1 - c_i^*$ and $1 - \omega_{c_i} \geq 1 - \omega_{c_i}^*$,

and then we have

$$\prod_{i=1}^{n} (1 - c_i)^{\alpha_i} \geq \prod_{i=1}^{n} \left(1 - c_i^*\right)^{\alpha_i} \text{ and } \prod_{i=1}^{n} \left(1 - \omega_{c_i}\right)^{\alpha_i}$$

$$\geq \prod_{i=1}^{n} \left(1 - \omega_{c_i}^*\right)^{\alpha_i}$$

since $\alpha_i \in [0, 1]$. Thus, we can write

$$1 - \prod_{i=1}^{n} (1 - c_i)^{\alpha_i} \leq 1 - \prod_{i=1}^{n} \left(1 - c_i^*\right)^{\alpha_i} \text{ and } 1$$

$$- \prod_{i=1}^{n} \left(1 - \omega_{c_i}\right)^{\alpha_i} \leq 1 - \prod_{i=1}^{n} \left(1 - \omega_{c_i}^*\right)^{\alpha_i}$$

which end the first part of proof. For the reliability part of CFZNWA operator, we have $r_i \leq r_i^*$ and $\omega_{r_i} \leq \omega_{r_i}^*$ for all i. Let $f(r) = \frac{1-r}{1+r}$ and $g(\omega) = \frac{1-\omega}{1+\omega}$ where $r, \omega \in [0, 1]$. It is obvious that both f and g are decreasing functions. Therefore, if $r_i \leq r_i^*$ and $\omega_{r_i} \leq \omega_{r_i}^*$ for all $i = 1, 2, \ldots n$, then $f(r_i) \geq f\left(r_i^*\right)$ and $g(\omega_i) \geq g(\omega_i^*)$, i.e., $\frac{1-r_i}{1+r_i} \geq \frac{1-r_i^*}{1+r_i^*}$ and $\frac{1-\omega_i}{1+\omega_i} \geq \frac{1-\omega_i^*}{1+\omega_i^*}$. Using the weight vector α_i, we have $\left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i} \geq \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i}$ and $\left(\frac{1-\omega_i}{1+\omega_i}\right)^{\alpha_i} \geq \left(\frac{1-\omega_i^*}{1+\omega_i^*}\right)^{\alpha_i}$. Then, these expressions can be written as products for $i = 1, 2, \ldots n$ and we have

$$\prod_{i=1}^{n} \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i} \geq \prod_{i=1}^{n} \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i} \text{ and } \prod_{i=1}^{n} \left(\frac{1-\omega_i}{1+\omega_i}\right)^{\alpha_i}$$

$$\geq \prod_{i=1}^{n} \left(\frac{1-\omega_i^*}{1+\omega_i^*}\right)^{\alpha_i}.$$

Since the following operations are same for r_i and ω_i, we only show for r_i.

$$\prod_{i=1}^{n} \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i} \geq \prod_{i=1}^{n} \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i} \iff 1 + \prod_{i=1}^{n} \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}$$

$$\geq 1 + \prod_{i=1}^{n} \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i}$$

$$\iff \frac{1}{1 + \prod_{i=1}^{n} \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}}$$

$$\leq \frac{1}{1 + \prod_{i=1}^{n} \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i}}$$

$$\iff \frac{2}{1 + \prod_{i=1}^{n} \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}}$$

$$\leq \frac{2}{1 + \prod_{i=1}^{n} \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i}}$$

$$\iff \frac{2}{1 + \prod_{i=1}^{n} \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}} - 1$$

$$\leq \frac{2}{1 + \prod_{i=1}^{n} \left(\frac{1-r_i^*}{1+r_i^*}\right)^{\alpha_i}} - 1$$

$$\iff \frac{\prod_{i=1}^{n} (1 + r_i)^{\alpha_i} - \prod_{i=1}^{n} (1 - r_i)^{\alpha_i}}{\prod_{i=1}^{n} (1 + r_i)^{\alpha_i} + \prod_{i=1}^{n} (1 - r_i)^{\alpha_i}}$$

$$\leq \frac{\prod_{i=1}^{n} \left(1 + r_i^*\right)^{\alpha_i} - \prod_{i=1}^{n} \left(1 - r_i^*\right)^{\alpha_i}}{\prod_{i=1}^{n} \left(1 + r_i^*\right)^{\alpha_i} + \prod_{i=1}^{n} \left(1 - r_i^*\right)^{\alpha_i}}$$

which end the second part of proof. Thus, from the first and second part of the proofs we have

$$CFZNWA(z_1, z_2, \ldots, z_n) \leq CFZNWA\left(z_1^*, z_2^*, \ldots, z_n^*\right)$$

Property 3 (Boundedness) Let $z_i = \langle\left(c_i, \omega_{c_i}\right), \left(r_i, \omega_{r_i}\right)\rangle$ be a collection of CFZNs where $i = 1, 2, \ldots, n$ and.

$$z^- = \left\langle\left(\min_i c_i, \min_i \omega_{c_i}\right), \left(\min_i r_i, \min_i \omega_{r_i}\right)\right\rangle$$

$$z^+ = \left\langle\left(\max_i c_i, \max_i \omega_{c_i}\right), \left(\max_i r_i, \max_i \omega_{r_i}\right)\right\rangle$$

Then,

$$z^- \leq CFZNWA(z_1, z_2, \ldots, z_n) \leq z^+ \tag{19}$$

Proof We prove for c_i and r_i, since the operations are same for ω_{c_i} and ω_{r_i}, respectively. Let $CFZNWA(z_1, z_2, \ldots, z_n) = \langle(c, \omega_c), (r, \omega_r)\rangle$. To avoid confusion caused by the complexity of the equations and ensure its fluency, we divide the proof into two parts, restriction and reliability, as we did in Property 2. For the first part, we have $\min_i c_i \leq c_i \leq \max_i c_i$ and $\min_i \omega_{c_i} \leq \omega_{c_i} \leq \max_i \omega_{c_i}$. Since the following operations are same for c_i and ω_{c_i}, we only show for c_i. Then,

$$\iff 1 - \min_i c_i \geq 1 - c_i \geq 1 - \max_i c_i \iff \left(1 - \min_i c_i\right)^{\alpha_i}$$

$$\geq (1 - c_i)^{\alpha_i} \geq \left(1 - \max_i c_i\right)^{\alpha_i} \iff \prod_{i=1}^{n} \left(1 - \min_i c_i\right)^{\alpha_i}$$

$$\geq \prod_{i=1}^{n} (1 - c_i)^{\alpha_i} \geq \prod_{i=1}^{n} \left(1 - \max_i c_i\right)^{\alpha_i}$$

$$\iff \left(1 - \min_i c_i\right)^{\sum_{i=1}^{n} \alpha_i} \geq \prod_{i=1}^{n} (1 - c_i)^{\alpha_i}$$

$$\geq \left(1 - \max_i c_i\right)^{\sum_{i=1}^{n} \alpha_i} \iff 1 - \min_i c_i \geq \prod_{i=1}^{n} (1 - c_i)^{\alpha_i}$$

$$\geq 1 - \max_i c_i \iff \min_i c_i \leq 1 - \prod_{i=1}^{n} (1 - c_i)^{\alpha_i} \geq \max_i c_i$$

which ends the first part of the proof. For the reliability part, let $f(r) = \frac{1-r}{1+r}$ and $g(\omega) = \frac{1-\omega}{1+\omega}$ where $r, \omega \in [0, 1]$. It is obvious that both f and g are decreasing functions. Since $\min_i r_i \leq r_i \leq \max_i r_i$ and $\min_i \omega_{r_i} \leq \omega_{r_i} \leq \max_i \omega_{r_i}$ for all i, then we have $f\left(\min_i r_i\right) \geq f(r_i) \geq f\left(\max_i r_i\right)$ and $g\left(\min_i \omega_{r_i}\right) \geq g(\omega_{r_i}) \geq g\left(\max_i \omega_{r_i}\right)$ which can also be written as:

$$\frac{1 - \min_i r_i}{1 + \min_i r_i} \geq \frac{1 - r_i}{1 + r_i} \geq \frac{1 - \max_i r_i}{1 + \max_i r_i} \text{ and } \frac{1 - \min_i \omega_{r_i}}{1 + \min_i \omega_{r_i}}$$

$$\geq \frac{1 - \omega_{r_i}}{1 + \omega_{r_i}} \geq \frac{1 - \max_i \omega_{r_i}}{1 + \max_i \omega_{r_i}}$$

Since the following operations are same for r_i and ω_i, we only show for r_i. Using the weight vector α_i, we have

$$\left(\frac{1 - \min_i r_i}{1 + \min_i r_i}\right)^{\alpha_i} \geq \left(\frac{1 - r_i}{1 + r_i}\right)^{\alpha_i} \geq \left(\frac{1 - \max_i r_i}{1 + \max_i r_i}\right)^{\alpha_i}$$

$$\Leftrightarrow \prod_{i=1}^n \left(\frac{1 - \min_i r_i}{1 + \min_i r_i}\right)^{\alpha_i}$$

$$\geq \prod_{i=1}^n \left(\frac{1 - r_i}{1 + r_i}\right)^{\alpha_i} \geq \prod_{i=1}^n \left(\frac{1 - \max_i r_i}{1 + \max_i r_i}\right)^{\alpha_i}$$

$$\Leftrightarrow \left(\frac{1 - \min_i r_i}{1 + \min_i r_i}\right)^{\sum_{i=1}^n \alpha_i} \geq \prod_{i=1}^n \left(\frac{1 - r_i}{1 + r_i}\right)^{\alpha_i}$$

$$\geq \left(\frac{1 - \max_i r_i}{1 + \max_i r_i}\right)^{\sum_{i=1}^n \alpha_i} \Leftrightarrow \frac{1 - \min_i r_i}{1 + \min_i r_i}$$

$$\geq \prod_{i=1}^n \left(\frac{1 - r_i}{1 + r_i}\right)^{\alpha_i} \geq \frac{1 - \max_i r_i}{1 + \max_i r_i}$$

$$\Leftrightarrow \frac{1}{1 + \frac{1 - \min_i r_i}{1 + \min_i r_i}} \leq \frac{1}{1 + \prod_{i=1}^n \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}}$$

$$\leq \frac{1}{1 + \frac{1 - \max_i r_i}{1 + \max_i r_i}} \Leftrightarrow \frac{1 + \min_i r_i}{2}$$

$$\leq \frac{1}{1 + \prod_{i=1}^n \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}} \leq \frac{1 + \max_i r_i}{2}$$

$$\Leftrightarrow \min_i r_i \leq \frac{2}{1 + \prod_{i=1}^n \left(\frac{1-r_i}{1+r_i}\right)^{\alpha_i}} - 1$$

$$\leq \max_i r_i \Leftrightarrow \min_i r_i$$

$$\leq \frac{\prod_{i=1}^n (1 + r_i)^{\alpha_i} - \prod_{i=1}^n (1 - r_i)^{\alpha_i}}{\prod_{i=1}^n (1 + r_i)^{\alpha_i} + \prod_{i=1}^n (1 - r_i)^{\alpha_i}}$$

$$\leq \max_i r_i$$

which end the second part of proof. Thus, from the first and second part of the proofs we have

$$z^- \leq CFZNWA(z_1, z_2, \ldots, z_n) \leq z^+$$

Definition 12 Let $z_i (i = 1, 2, \ldots, n)$ be a collection of CFZNs. A complex fuzzy Z-number weighted geometric (CFZNWG) operator is a mapping $CFZNWG : \mathcal{Z}^n \to \mathcal{Z}$ defined by.

$$CFZNWG(z_1, z_2, \ldots, z_n)$$
$$= \otimes_{i=1}^n (z_i)^{\alpha_i}$$
$$= z_1^{\alpha_1} \otimes z_2^{\alpha_2} \otimes \ldots \otimes z_n^{\alpha_n} \qquad (20)$$

where $\alpha_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the weight vector of z_i with $\alpha_i \in [0, 1]$ and $\sum_{i=1}^n \alpha_i = 1$. If the weight vector is taken as $\alpha_i = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, the CFZNWG operator reduces to complex fuzzy Z-number geometric (CFZNG) operator as

$$CFZNG(z_1, z_2, \ldots, z_n) = \otimes_{i=1}^n z_i.$$

Theorem 5 Let $z_i (i = 1, 2, \ldots, n)$ be a collection of CFZNs. Then, the CFZNWG operator is also a CFZN and can be expressed as

$$CFZNWG(z_1, z_2, \ldots, z_n)$$
$$= \left\langle \left(\prod_{i=1}^n (c_i)^{\alpha_i}, \prod_{i=1}^n \left(\omega_{c_i}\right)^{\alpha_i}\right), \left(\frac{2\prod_{i=1}^n (r_i)^{\alpha_i}}{\prod_{i=1}^n (2-r_i)^{\alpha_i} + \prod_{i=1}^n (r_i)^{\alpha_i}}, \frac{2\prod_{i=1}^n \left(\omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^n \left(2-\omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^n \left(\omega_{r_i}\right)^{\alpha_i}}\right) \right\rangle$$
$$\qquad (21)$$

Proof It is similar to proof of theorem 4 given in Eq. (16).

The introduced $CFZNWG$ operator also holds the same specifications. To show these features and avoid repetition, let $z_i (i = 1, 2, \ldots, n)$ be a collection of CFZNs and $\alpha_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the weight vector of z_i with $\alpha_i \in [0, 1]$ and $\sum_{i=1}^n \alpha_i = 1$.

Property 1 (Idempotency). If all z_i are equal, i.e., $z_i = z = \langle (c, \omega_c), (r, \omega_r) \rangle$ for all i, then

$$CFZNWG(z_1, z_2, \ldots, z_n) = z^\alpha \otimes z^\alpha \otimes \ldots \otimes z^\alpha = z.$$
$$\qquad (22)$$

Property 2 (Monotonicity). Let $z_i = \langle (c_i, \omega_{c_i}), (r_i, \omega_{r_i}) \rangle$ and $z_i^* = \langle (c_i^*, \omega_{c_i}^*), (r_i^*, \omega_{r_i}^*) \rangle$ be two collection of CFZNs

where $i = 1, 2, \ldots n$. $z_i \leq z_i^*$ for all i, i.e., $c_i \leq c_i^*$, $r_i \leq r_i^*$, $\omega_{c_i} \leq \omega_{c_i}^*$ and $\omega_{r_i} \leq \omega_{r_i}^*$, then.

$$CFZNWG(z_1, z_2, \ldots, z_n) \leq CFZNWG\left(z_1^*, z_2^*, \ldots, z_n^*\right). \tag{23}$$

Property 3 (Boundedness) Let $z_i = \langle (c_i, \omega_{c_i}), (r_i, \omega_{r_i}) \rangle$ be a collection of CFZNs where $i = 1, 2, \ldots, n$ and

$$z^- = \left\langle \left(\min_i c_i, \min_i \omega_{c_i} \right), \left(\min_i r_i, \min_i \omega_{r_i} \right) \right\rangle z^+$$
$$= \left\langle \left(\max_i c_i, \max_i \omega_{c_i} \right), \left(\max_i r_i, \max_i \omega_{r_i} \right) \right\rangle$$

Then,

$$z^- \leq CFZNWG(z_1, z_2, \ldots, z_n) \leq z^+ \tag{24}$$

These properties of the CFZNWG aggregation operator can be easily proved using the techniques in the proof of properties of the CFZNWA aggregation operator.

Example 3 Let there are four CFZNs $z_1 = \langle (0.6, 0.5), (0.4, 0.3) \rangle$, $z_2 = \langle (0.8, 0.3), (0.5, 0.2) \rangle$, $z_3 = \langle (0.4, 0.2), (0.4, 0.5) \rangle$ and $z_4 = \langle (0.5, 0.1), (0.8, 0.3) \rangle$, and $\alpha = (0.3, 0.4, 0.2, 0.1)$ be their weight vector. Using the Eq. (16) for CFZNWA operator we compute:

$$1 - \prod_{i=1}^4 (1 - c_i)^{\alpha_i}$$
$$= 1 - (1 - 0.6)^{0.3} \times (1 - 0.8)^{0.4}$$
$$\times (1 - 0.4)^{0.2} \times (1 - 0.5)^{0.1}$$
$$= 0.66381 - \prod_{i=1}^4 \left(1 - \omega_{c_i}\right)^{\alpha_i} = 1 - (1 - 0.5)^{0.3}$$
$$\times (1 - 0.3)^{0.4} \times (1 - 0.2)^{0.2} \times (1 - 0.1)^{0.1}$$
$$= 0.3335 \frac{\prod_{i=1}^4 (1 + r_i)^{\alpha_i} - \prod_{i=1}^4 (1 - r_i)^{\alpha_i}}{\prod_{i=1}^4 (1 + r_i)^{\alpha_i} + \prod_{i=1}^4 (1 - r_i)^{\alpha_i}}$$
$$= \frac{1.4758 - 0.4998}{1.4758 + 0.4998} = \frac{0.9760}{1.9756}$$
$$= 0.4940 \frac{\prod_{i=1}^n \left(1 + \omega_{r_i}\right)^{\alpha_i} - \prod_{i=1}^n \left(1 - \omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^n \left(1 + \omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^n \left(1 - \omega_{r_i}\right)^{\alpha_i}}$$
$$= \frac{1.2956 - 0.6903}{1.2956 + 0.6903} = \frac{0.6053}{1.9859} = 0.3048$$

Therefore, we get:

$$CFZNWA(z_1, z_2, z_3, z_4) = \langle (0.6638, 0.3335), (0.4940, 0.3048) \rangle$$

It is obvious that the result is also CFZN. Similarly, using the Eq. (21) for CFZNWG operator, we compute:

$$\prod_{i=1}^4 (c_i)^{\alpha_i} = 0.6^{0.3} \times 0.8^{0.4} \times 0.4^{0.2} \times 0.5^{0.1}$$
$$= 0.6095 \prod_{i=1}^4 (\omega_i)^{\alpha_i}$$
$$= 0.5^{0.3} \times 0.3^{0.4} \times 0.2^{0.2} \times 0.1^{0.1}$$
$$= 0.2889 \frac{2 \prod_{i=1}^n (r_i)^{\alpha_i}}{\prod_{i=1}^n (2 - r_i)^{\alpha_i} + \prod_{i=1}^n (r_i)^{\alpha_i}}$$
$$= \frac{2 \times 0.4687}{1.5150 + 0.4687}$$
$$= 0.4726 \frac{2 \prod_{i=1}^n \left(\omega_{r_i}\right)^{\alpha_i}}{\prod_{i=1}^n \left(2 - \omega_{r_i}\right)^{\alpha_i} + \prod_{i=1}^n \left(\omega_{r_i}\right)^{\alpha_i}}$$
$$= \frac{2 \times 0.2825}{1.6963 + 0.2825} = 0.2855$$

Therefore, we get:

$$CFZNWG(z_1, z_2, z_3, z_4) = \langle (0.6095, 0.2889), (0.4726, 0.2855) \rangle$$

We can see that the result of CFZNWG operator is also a CFZN.

Numerical application using CFZNWA and CFZNWG operators

In this section, we present a MCDM approach based on CFZNWA and CFZNWG operators under the CFZN environment. Suppose there are m alternatives $A = \{A_1, A_2, \ldots, A_m\}$ and n criteria $C = \{C_1, C_2, \ldots, C_n\}$ for a MCDM problem which are in the form of CFNZs. The set of criteria C is divided into two discrete set which are called the benefit set I and the cost set J where $I \cap J = \varnothing$ and $I \cup J = C$. Let $\alpha_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the weight vector of the n criteria with $\alpha_i \in [0, 1]$ and $\sum_{i=1}^n \alpha_i = 1$. Then, the steps of the constructed MCDM are given as follows:

Step 1. Obtain the evolution information under CFZNs and construct the decision matrix $D(z_i)_{m \times n}$:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \cdots \\ A_m \end{array} \begin{array}{c} C_1 \quad C_2 \ldots C_n \\ \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix} \end{array}$$

Step 2. Normalize the decision matrix $D(z_i)_{m \times n}$ using the following equation:

$$N_{ij} = \begin{cases} \langle (c_i, \omega_{c_i}), (r_i, \omega_{r_i}) \rangle, & if\, C_j \in I \\ \langle (1 - c_i, \omega_{c_i}), (1 - r_i, \omega_{r_i}) \rangle, & if\, C_j \in J \end{cases}$$

Step 3. Aggregate the normalized decision matrix N based on CFZNWA or CFZNWG aggregation operators with a given weight vector using the following equations:

$$R_i = CFZNWA(z_{i1}, z_{i2}, \ldots, z_{in}) \tag{25}$$
$$= \left\langle \begin{pmatrix} \left(1 - \prod_{j=1}^{n} (1 - c_j)^{\alpha_j}, 1 - \prod_{j=1}^{n} (1 - \omega_{c_j})^{\alpha_j}\right), \\ \left(\frac{\prod_{j=1}^{n} (1+r_j)^{\alpha_j} - \prod_{j=1}^{n} (1-r_j)^{\alpha_j}}{\prod_{j=1}^{n} (1+r_j)^{\alpha_j} + \prod_{j=1}^{n} (1-r_j)^{\alpha_j}}, \frac{\prod_{j=1}^{n} \left(1+\omega_{r_j}\right)^{\alpha_j} - \prod_{j=1}^{n} \left(1-\omega_{r_j}\right)^{\alpha_j}}{\prod_{j=1}^{n} \left(1+\omega_{r_j}\right)^{\alpha_j} + \prod_{j=1}^{n} \left(1-\omega_{r_j}\right)^{\alpha_j}}\right) \end{pmatrix} \right\rangle$$

$$R_i = CFZNWG(z_{i1}, z_{i2}, \ldots, z_{in}) \tag{26}$$
$$= \left\langle \begin{pmatrix} \left(\prod_{j=1}^{n} (c_j)^{\alpha_j}, \prod_{j=1}^{n} \left(\omega_{c_j}\right)^{\alpha_j}\right), \\ \left(\frac{2 \prod_{j=1}^{n} (r_j)^{\alpha_j}}{\prod_{j=1}^{n} (2-r_j)^{\alpha_j} + \prod_{j=1}^{n} (r_j)^{\alpha_j}}, \frac{2 \prod_{j=1}^{n} \left(\omega_{r_j}\right)^{\alpha_j}}{\prod_{j=1}^{n} \left(2-\omega_{r_j}\right)^{\alpha_j} + \prod_{j=1}^{n} \left(\omega_{r_j}\right)^{\alpha_j}}\right) \end{pmatrix} \right\rangle$$

Step 4. Calculate the score values of R_i for $i = 1, 2, \ldots, m$ using the score function as:

$$S(R_i) = \frac{1}{4}\left(c_i + \omega_{c_i}\right) \cdot \left(r_i + \omega_{r_i}\right) \tag{27}$$

If there are any equality, use the algorithm given in Eq. (14) to compare R_i values.

Step 5. Rank the alternatives.

A case study

Unmanned Aerial Vehicles (UAVs) are autonomous or remotely controlled aircrafts that can operate without a human pilot. These aircraft have gained significant popularity and applications in a variety of industries and sectors due to their versatility, manoeuvrability, and cost-effectiveness. UAVs are equipped with advanced technologies such as cameras, communication systems, and sensors to perform various tasks. While Unmanned Combat Aerial Vehicles (UCAVs) are mostly used for military operations, they can also be utilized for aerial photography and videography, surveillance and monitoring, search and rescue operations, agriculture and environmental monitoring, as well as transportation. Turkey, on the other hand, has made significant advancements in this field in recent years. An independent technology company is developing UAVs and UCAVs technology with the request and support of the Ministry of Defence and the Turkish Space Agency. As these developments continue, ongoing efforts are being made to enhance their capabilities, including increasing flight endurance, improving obstacle avoidance systems, and enabling autonomous decision-making. Therefore, there are numerous models available to cater to various

requirements. Additionally, countries that do not possess this technology seek to acquire these vehicles for defence purposes and other applications such as rescue missions. However, despite the independence of the UAV-producing company, it is improbable for such agreements to be reached without the approval of the Ministry of Defence, aligning with national interests. Consequently, during the sales phase, the Ministry of Defence has the authority to restrict the technologies included in the models to be sold. Because no country wants its newest technology to pass into the hands of any foreign institution. For this reason, it is very important for the purchasing institution to choose the best possible alternative.

In this example, we're using features from a model that a country's defence ministry want as criteria, based on the attributes in the examples used by Garg [5] and Jia et al. [12]. This country's defence department determines 8 software criteria for the model it wants to buy. These criteria are C_1 : flight range, C_2 : measurement accuracy, C_3 : anti-interference performance, C_4 : image processing capability, C_5 : generation of contour lines using DEM/DSM, C_6 : generation of 3D modelling/texturing capabilities C_7 : measurement tools for co-ordinate/distance/area/volume and C_8 : price. After receiving the opinion of the Ministry of Defence, 6 alternatives are determined for sale after some restrictions. These restrictions include criteria features and software versions. In addition, since technologies will develop over the years, how much of the limitation of the software will be removed and whether version updates can be obtained are important possibilities that may be encountered. Since all these situations must be considered at the same time and decided at once, it becomes very critical to be able to express this problem with numerical values. This is where the importance of CFZNs comes into play. CFZNs can express these 4 inputs at the same time, while simultaneously considering constraints and probabilities. The evolution values for A_1 at C_1 are $\langle (0.5, 0.4), (0.4, 0.6) \rangle$ which represents that the seller cut down feature of C_1 for fifty percent, and the version for forty percent. In addition, we express the reliability of the software and update constraint will be removed by forty percent and sixty percent, respectively in coming years. Additionally, weights of the criteria are given as $\alpha = (0.12\,0.08\,0.09\,0.13\,0.08\,0.11\,0.10\,0.14\,0.15)$ by the expert of the purchasing country.

Step 1. The evolution values given by expert in terms of CFZNs are represented in Table 1 as the decision matrix $D(z_i)_{m \times n}$.

Step 2. Normalize the decision matrix $D(z_i)_{m \times n}$ given in Table 2.

Since the only cost criteria is price of the UAVs, we have the normalized decision matrix N_{ij} as:

Table 1 Assessments of the expert

	C_1	C_2	C_3	C_4
A_1	$\langle(0.1, 0.2), (0.5, 0.8)\rangle$	$\langle(0.7, 0.2), (0.1, 0.7)\rangle$	$\langle(0.5, 0.5), (0.3, 0.3)\rangle$	$\langle(0.9, 0.5), (0.7, 0.2)\rangle$
A_2	$\langle(0.3, 0.3), (0.5, 0.6)\rangle$	$\langle(0.8, 0.6), (0.1, 0.8)\rangle$	$\langle(0.3, 0.8), (0.7, 0.6)\rangle$	$\langle(0.4, 0.1), (0.9, 0.6)\rangle$
A_3	$\langle(0.5, 0.3), (0.4, 0.3)\rangle$	$\langle(0.3, 0.2), (0.2, 0.5)\rangle$	$\langle(0.1, 0.9), (0.2, 0.2)\rangle$	$\langle(0.7, 0.7), (0.4, 0.1)\rangle$
A_4	$\langle(0.3, 0.3), (0.1, 0.9)\rangle$	$\langle(0.5, 0.5), (0.5, 0.4)\rangle$	$\langle(0.3, 0.9), (0.5, 0.7)\rangle$	$\langle(0.7, 0.8), (0.3, 0.3)\rangle$
A_5	$\langle(0.9, 0.8), (0.9, 0.5)\rangle$	$\langle(0.2, 0.5), (0.7, 0.7)\rangle$	$\langle(0.8, 0.7), (0.6, 0.3)\rangle$	$\langle(0.4, 0.8), (0.4, 0.1)\rangle$
A_6	$\langle(0.7, 0.8), (0.8, 0.2)\rangle$	$\langle(0.6, 0.7), (0.7, 0.6)\rangle$	$\langle(0.1, 0.2), (0.3, 0.6)\rangle$	$\langle(0.2, 0.1), (0.9, 0.2)\rangle$
	C_5	C_6	C_7	C_8
A_1	$\langle(0.1, 0.2), (0.6, 0.6)\rangle$	$\langle(0.9, 0.1), (0.2, 0.4)\rangle$	$\langle(0.3, 0.1), (0.8, 0.4)\rangle$	$\langle(0.4, 0.6), (0.9, 0.7)\rangle$
A_2	$\langle(0.3, 0.1), (0.3, 0.9)\rangle$	$\langle(0.2, 0.5), (0.9, 0.2)\rangle$	$\langle(0.5, 0.6), (0.6, 0.6)\rangle$	$\langle(0.7, 0.4), (0.2, 0.8)\rangle$
A_3	$\langle(0.1, 0.4), (0.8, 0.5)\rangle$	$\langle(0.1, 0.8), (0.6, 0.5)\rangle$	$\langle(0.5, 0.5), (0.6, 0.6)\rangle$	$\langle(0.8, 0.8), (0.8, 0.5)\rangle$
A_4	$\langle(0.7, 0.2), (0.8, 0.2)\rangle$	$\langle(0.9, 0.7), (0.2, 0.8)\rangle$	$\langle(0.9, 0.6), (0.4, 0.4)\rangle$	$\langle(0.3, 0.3), (0.9, 0.2)\rangle$
A_5	$\langle(0.8, 0.4), (0.2, 0.1)\rangle$	$\langle(0.6, 0.4), (0.4, 0.4)\rangle$	$\langle(0.5, 0.3), (0.5, 0.1)\rangle$	$\langle(0.8, 0.3), (0.3, 0.6)\rangle$
A_6	$\langle(0.9, 0.6), (0.3, 0.5)\rangle$	$\langle(0.7, 0.3), (0.2, 0.5)\rangle$	$\langle(0.2, 0.3), (0.9, 0.3)\rangle$	$\langle(0.1, 0.4), (0.9, 0.7)\rangle$

Step 3. Aggregate the normalized decision matrix N_{ij} based on CFZNWA or CFZNWG aggregation operators given in Eqs. (25) or (26), respectively, using given weights.

Using the CFZNWA given in Eq. (25), the aggregated values R_i are obtained as

$R_1 = \langle(0.5919, 0.3061), (0.3937, 0.4597)\rangle$

$R_2 = \langle(0.3575, 0.4069), (0.6371, 0.6286)\rangle$

$R_3 = \langle(0.3285, 0.6082), (0.3818, 0.3323)\rangle$

$R_4 = \langle(0.6449, 0.5536), (0.3025, 0.6134)\rangle$

$R_5 = \langle(0.5664, 0.5271), (0.5390, 0.3256)\rangle$

$R_6 = \langle(0.6005, 0.4190), (0.5572, 0.3339)\rangle$

Step 4. Calculate the score values of R_i for $i = 1, 2, \ldots, m$ using the score function.

Using the score function given in Eq. (27) we have $S(R_1) = 0.1916$, $S(R_2) = 0.2419$, $S(R_3) = 0.1672$, $S(R_4) = 0.2744$, $S(R_5) = 0.2363$ and $S(R_6) = 0.2271$. Therefore, we have the ranking $A_4 > A_2 > A_5 > A_6 > A_1 > A_3$ and we conclude that A_4 is the best and A_3 is the worst alternative. Visual interpretation can be seen in Fig. 1.

Validity test of the proposed method

An important problem with decision-making methods that rank alternatives in real-life problems in terms of criteria is that sometimes these methods can provide different answers (orders) when used with the same numerical data. Therefore, the issue arises of how to evaluate the performance of such methods. Since each decision-making method has relative advantages and disadvantages, it is almost impossible to determine both which decision-making method is consistent, and which is the best alternative for a given decision problem. Therefore, the decision-making methods used or

Table 2 Normalized decision matrix N_{ij}

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{A}_1$	$\langle(0.1, 0.2), (0.5, 0.8)\rangle$	$\langle(0.7, 0.2), (0.1, 0.7)\rangle$	$\langle(0.5, 0.5), (0.3, 0.3)\rangle$	$\langle(0.9, 0.5), (0.7, 0.2)\rangle$
$\mathcal{A}_2$	$\langle(0.3, 0.3), (0.5, 0.6)\rangle$	$\langle(0.8, 0.6), (0.1, 0.8)\rangle$	$\langle(0.3, 0.8), (0.7, 0.6)\rangle$	$\langle(0.4, 0.1), (0.9, 0.6)\rangle$
$\mathcal{A}_3$	$\langle(0.5, 0.3), (0.4, 0.3)\rangle$	$\langle(0.3, 0.2), (0.2, 0.5)\rangle$	$\langle(0.1, 0.9), (0.2, 0.2)\rangle$	$\langle(0.7, 0.7), (0.4, 0.1)\rangle$
$\mathcal{A}_4$	$\langle(0.3, 0.3), (0.1, 0.9)\rangle$	$\langle(0.5, 0.5), (0.5, 0.4)\rangle$	$\langle(0.3, 0.9), (0.5, 0.7)\rangle$	$\langle(0.7, 0.8), (0.3, 0.3)\rangle$
$\mathcal{A}_5$	$\langle(0.9, 0.8), (0.9, 0.5)\rangle$	$\langle(0.2, 0.5), (0.7, 0.7)\rangle$	$\langle(0.8, 0.7), (0.6, 0.3)\rangle$	$\langle(0.4, 0.8), (0.4, 0.1)\rangle$
$\mathcal{A}_6$	$\langle(0.7, 0.8), (0.8, 0.2)\rangle$	$\langle(0.6, 0.7), (0.7, 0.6)\rangle$	$\langle(0.1, 0.2), (0.3, 0.6)\rangle$	$\langle(0.2, 0.1), (0.9, 0.2)\rangle$

	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle(0.1, 0.2), (0.6, 0.6)\rangle$	$\langle(0.9, 0.1), (0.2, 0.4)\rangle$	$\langle(0.3, 0.1), (0.8, 0.4)\rangle$	$\langle(0.6, 0.6), (0.1, 0.7)\rangle$
$\mathcal{A}_2$	$\langle(0.3, 0.1), (0.3, 0.9)\rangle$	$\langle(0.2, 0.5), (0.9, 0.2)\rangle$	$\langle(0.5, 0.6), (0.6, 0.6)\rangle$	$\langle(0.3, 0.4), (0.8, 0.8)\rangle$
$\mathcal{A}_3$	$\langle(0.1, 0.4), (0.8, 0.5)\rangle$	$\langle(0.1, 0.8), (0.6, 0.5)\rangle$	$\langle(0.5, 0.5), (0.6, 0.6)\rangle$	$\langle(0.2, 0.8), (0.2, 0.5)\rangle$
$\mathcal{A}_4$	$\langle(0.7, 0.2), (0.8, 0.2)\rangle$	$\langle(0.9, 0.7), (0.2, 0.8)\rangle$	$\langle(0.9, 0.6), (0.4, 0.4)\rangle$	$\langle(0.7, 0.3), (0.1, 0.2)\rangle$
$\mathcal{A}_5$	$\langle(0.8, 0.4), (0.2, 0.1)\rangle$	$\langle(0.6, 0.4), (0.4, 0.4)\rangle$	$\langle(0.5, 0.3), (0.5, 0.1)\rangle$	$\langle(0.2, 0.3), (0.7, 0.6)\rangle$
$\mathcal{A}_6$	$\langle(0.9, 0.6), (0.3, 0.5)\rangle$	$\langle(0.7, 0.3), (0.2, 0.5)\rangle$	$\langle(0.2, 0.3), (0.9, 0.3)\rangle$	$\langle(0.9, 0.4), (0.1, 0.7)\rangle$

Fig. 1 Ranking of alternatives
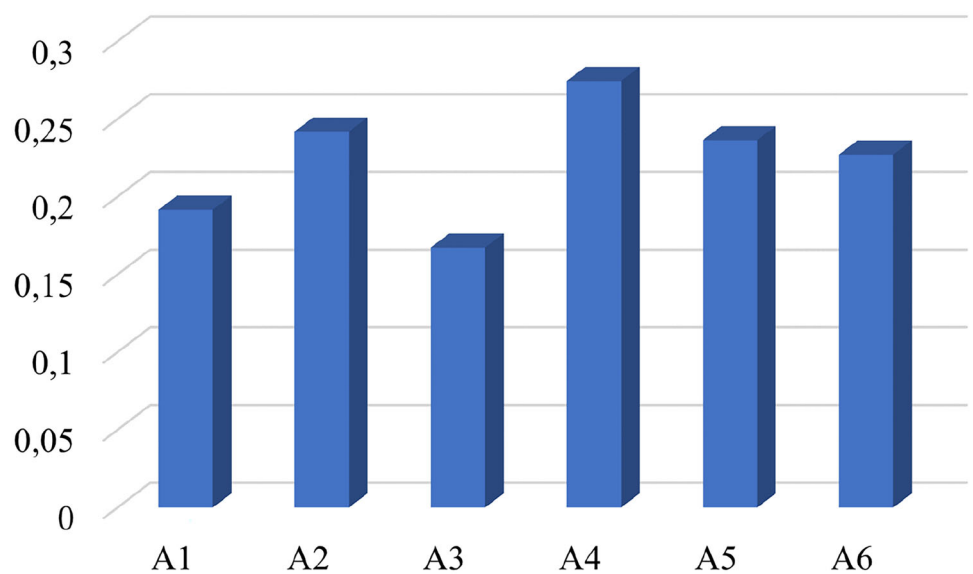
developed need to undergo testing procedures. These procedures were first introduced by Triantaphyllou [39] and are still used extensively today to measure the consistency and validity of newly developed methods. These test criteria are:

1. When a suboptimal alternative is replaced with a worse alternative without changing the importance of each decision criterion, the ranking of the best alternative should remain the same (This criterion is interpreted as "An effective decision-making method should avoid the rank reversal phenomenon effectively when an alternative is added or deleted" in some studies).
2. Decision making method should satisfy the transitive property.
3. When a decision-making problem is decomposed into smaller problems and a decision-making method is applied to rank the alternatives for these smaller problems, the combined ranking of the alternatives should be the same as the original ranking of the problem.

Validity test for criterion 1: If we replace $\mathcal{A}_6$ with $\mathcal{A}_1$ and the steps are executed, since aggregation operators are row-based operations, nothing changes except the ranking of $\mathcal{A}_5$ and $\mathcal{A}_1$ as $\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_6 > \mathcal{A}_3$.

Validity test for criterion 2 and 3: Let separate the problem into three sub-problems taking the alternatives randomly as $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_6\}$, $\{\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5, \mathcal{A}_6\}$ and $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5\}$. Then, the rankings are obtained as $\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_6 > \mathcal{A}_1$, $\mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_3$ and $\mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_3$, respectively. If we combine the ranking, we have $\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_1 > \mathcal{A}_3$ which is the original one. Additionally, it is easy to see that if $\mathcal{A}_4 > \mathcal{A}_2$ and $\mathcal{A}_2 > \mathcal{A}_3$ from sub-problems, then we have $\mathcal{A}_4 > \mathcal{A}_3$ which is an example of transitive property.

Comparison and discussion

The proposed CFZNS is a novel set theory that emerges complex fuzzy sets and Z-numbers. Besides, it should also be noted that CFZNS is a generalization of the Z-numbers. Although the topic of Z-number has been trending in recent years, there have been few studies in this field concerning arithmetic operations. These studies have either treated the reliability component of Z-number as a linguistic variable or converted the Z-number to crisp numbers using a limited set of operations. Alternatively, some studies have treated the reliability operations the same as the constraint operations. These situations can lead to a loss of knowledge when applied to real-life problems. This is because not all arithmetic operations can be accurately represented using linguistic operations, and operations are often carried out by converting Z-numbers to crisp numbers. The conversion of

information to crisp number results in a loss of information. In the light of this information, there are few cases where we can compare the proposed CFZNS, since the proposed set and its operations are also new. Therefore, we compare the results with CFZNWG operator, Z-numbers, Z-numbers with same arithmetic operations and complex fuzzy sets.

I. We use CFZNWG operator given in (26) to aggregate decision matrix $D(z_i)_{m \times n}$ given in Table 1. The aggregated values R_i are evaluated as

$$R_1 = \langle (0.5919, 0.3061), (0.3937, 0.4597) \rangle$$
$$R_2 = \langle (0.3575, 0.4069), (0.6371, 0.6286) \rangle$$
$$R_3 = \langle (0.3285, 0.6082), (0.3818, 0.3323) \rangle$$
$$R_4 = \langle (0.6449, 0.5536), (0.3025, 0.6134) \rangle$$
$$R_5 = \langle (0.5664, 0.5271), (0.5390, 0.3256) \rangle$$
$$R_6 = \langle (0.6005, 0.4190), (0.5572, 0.3339) \rangle$$

and the score values are obtained as $S(R_1) = 0.1823$, $S(R_2) = 0.2601$, $S(R_3) = 0.1932$, $S(R_4) = 0.2798$, $S(R_5) = 0.2550$ and $S(R_6) = 0.2040$ using score function given in Eq. (27). Therefore, we have the ranking $\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_3 > \mathcal{A}_1$ and we conclude that $\mathcal{A}_4$ is the best alternative. This ranking result is consistent with the result operated by the CFZNWA operator, as can be seen in Table 3. This shows how compatible and logical the arithmetic operations given in Eq. (11) are.

II. By taking $\omega_c = 0$ and $\omega_r = 0$, decision matrix reduces to Z-numbers which has the form $z = \langle \mathbb{c}, \mathbb{r} \rangle$, and the decision matrix becomes.

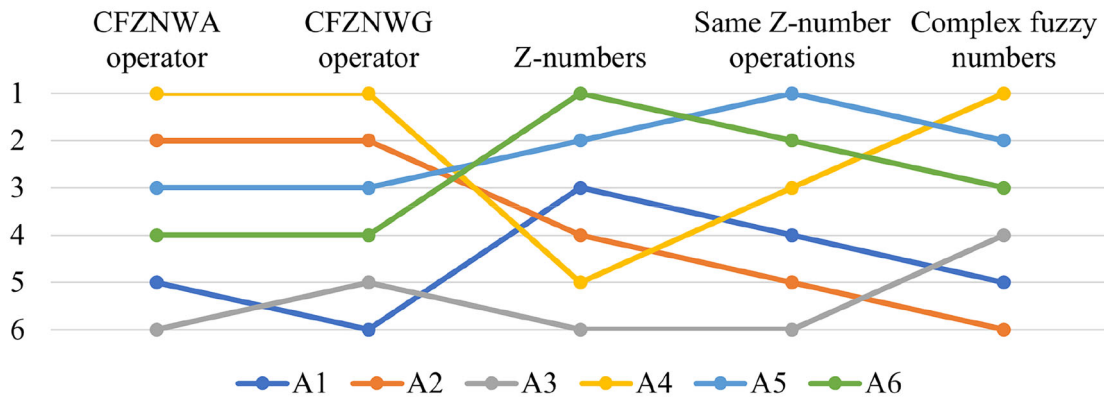|	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{A}_1$	$\langle 0.1, 0.5 \rangle$	$\langle 0.7, 0.1 \rangle$	$\langle 0.5, 0.3 \rangle$	$\langle 0.9, 0.7 \rangle$
$\mathcal{A}_2$	$\langle 0.3, 0.5 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.3, 0.7 \rangle$	$\langle 0.4, 0.9 \rangle$
$\mathcal{A}_3$	$\langle 0.5, 0.4 \rangle$	$\langle 0.3, 0.2 \rangle$	$\langle 0.1, 0.2 \rangle$	$\langle 0.7, 0.4 \rangle$
$\mathcal{A}_4$	$\langle 0.3, 0.1 \rangle$	$\langle 0.5, 0.5 \rangle$	$\langle 0.3, 0.5 \rangle$	$\langle 0.7, 0.3 \rangle$
$\mathcal{A}_5$	$\langle 0.9, 0.9 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.8, 0.6 \rangle$	$\langle 0.4, 0.4 \rangle$
$\mathcal{A}_6$	$\langle 0.7, 0.8 \rangle$	$\langle 0.6, 0.7 \rangle$	$\langle 0.1, 0.3 \rangle$	$\langle 0.2, 0.9 \rangle$
	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle 0.1, 0.6 \rangle$	$\langle 0.9, 0.2 \rangle$	$\langle 0.3, 0.8 \rangle$	$\langle 0.4, 0.9 \rangle$
$\mathcal{A}_2$	$\langle 0.3, 0.3 \rangle$	$\langle 0.2, 0.9 \rangle$	$\langle 0.5, 0.6 \rangle$	$\langle 0.7, 0.2 \rangle$
$\mathcal{A}_3$	$\langle 0.1, 0.8 \rangle$	$\langle 0.1, 0.6 \rangle$	$\langle 0.5, 0.6 \rangle$	$\langle 0.8, 0.8 \rangle$
$\mathcal{A}_4$	$\langle 0.7, 0.8 \rangle$	$\langle 0.9, 0.2 \rangle$	$\langle 0.9, 0.4 \rangle$	$\langle 0.3, 0.9 \rangle$
$\mathcal{A}_5$	$\langle 0.8, 0.2 \rangle$	$\langle 0.6, 0.4 \rangle$	$\langle 0.5, 0.5 \rangle$	$\langle 0.8, 0.3 \rangle$
$\mathcal{A}_6$	$\langle 0.9, 0.3 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.2, 0.9 \rangle$	$\langle 0.1, 0.9 \rangle$

Fig. 2 Ranking comparison of alternatives

Fig. 3 Ranking comparison of alternatives with radar plot
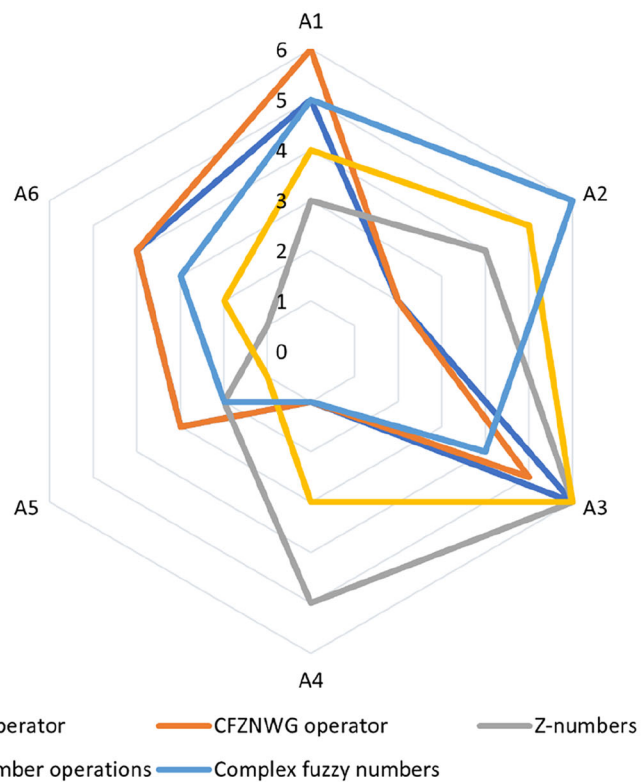
and updating CFZNWA operator given in Eq. (16) for Z-numbers as

$$ZNWA(z_1, z_2, \ldots, z_n)$$
$$= \left(1 - \prod_{i=1}^{n} (1 - c_i)^{\alpha_i}, \ \frac{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} - \prod_{i=1}^{n} (1-r_i)^{\alpha_i}}{\prod_{i=1}^{n} (1+r_i)^{\alpha_i} + \prod_{i=1}^{n} (1-r_i)^{\alpha_i}} \right)$$

we have

$$R_1 = \langle 0.5919, 0.3937 \rangle \, R_2 = \langle 0.3575, 0.6371 \rangle$$
$$R_3 = \langle 0.3285, 0.3818 \rangle \, R_4 = \langle 0.6449, 0.3025 \rangle$$
$$R_5 = \langle 0.5664, 0.5390 \rangle \, R_6 = \langle 0.6005, 0.5572 \rangle$$

Table 3 Comparative study

Methods	Rankings
CFZNWG Operator	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_3 > \mathcal{A}_1$
Z-numbers with ZNWA operator	$\mathcal{A}_6 > \mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3$
Z-numbers with same operations	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_6 > \mathcal{A}_1 > \mathcal{A}_5 > \mathcal{A}_3$
Complex fuzzy set	$\mathcal{A}_4 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_2$
Proposed approach	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_1 > \mathcal{A}_3$

Table 4 Sample of ıris data

	Sepal		Petal	
	Length	Width	Length	Width
Iris-setosa				
S_1	5.1	3.5	1.4	0.2
S_2	4.9	3.0	1.4	0.2
S_3	4.7	3.2	1.3	0.2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Iris-versicolor				
S_{51}	7.0	3.2	4.7	1.4
S_{52}	6.4	3.2	4.5	1.5
S_{53}	6.9	3.1	4.9	1.5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Iris-virginica				
S_{101}	6.3	3.3	6.0	2.5
S_{102}	5.8	2.7	5.1	1.9
S_{103}	7.1	3.0	5.9	2.1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Using the score function in Eq. (27) we have $S(R_1) = 0.1165$, $S(R_2) = 0.1139$, $S(R_3) = 0.0627$, $S(R_4) = 0.0975$, $S(R_5) = 0.1526$ and $S(R_6) = 0.1673$ and the ranking is $\mathcal{A}_6 > \mathcal{A}_5 > \mathcal{A}_1 > \mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3$. Looking at Table 3, this ranking result is quite different from the proposed approach. The best alternative, A_4, found with both CFZNWA and CFZNWG operators, falls to the lower ranks here, while the middle-ranked alternative A_6 is the best alternative here. This case shows how effective CFZNs, which are a generalization of Z-numbers, are in the results.

III. If we use same arithmetic operations calculating Z-number for restriction and reliability, we have

$$R_1 = \langle 0.4252, 0.4828\rangle \quad R_2 = \langle 0.6665, 0.6546\rangle$$

$$R_3 = \langle 0.4043, 0.3475\rangle \quad R_4 = \langle 0.3270, 0.6466\rangle$$

$$R_5 = \langle 0.5654, 0.3487\rangle \quad R_6 = \langle 0.6011, 0.3487\rangle$$

Using the score function we have $S(R_1) = 0.1027$, $S(R_2) = 0.2181$, $S(R_3) = 0.0702$, $S(R_4) = 0.1057$, $S(R_5) = 0.0986$ and $S(R_6) = 0.1048$ and the ranking is $\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_6 > \mathcal{A}_1 > \mathcal{A}_5 > \mathcal{A}_3$. Again, the ranking result is different from the proposed approach except the worst alternative from the Table 3. However, it is possible to see the changes of successive alternatives in these rankings such as A_2 and A_4. The best alternative has changed, although not as big changes as in other methods. This situation also a demonstration of how proposed approach effects the results.

IV. If we take only the first part of the proposed approach to see the impact of the reliability, the operations reduce to complex fuzzy sets which can be written as $A_{\mathbb{C}} = (\mathbb{c}, \omega_{\mathbb{c}})$ and we have.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{A}_1$	$\langle(0.1, 0.2)\rangle$	$\langle(0.7, 0.2)\rangle$	$\langle(0.5, 0.5)\rangle$	$\langle(0.9, 0.5)\rangle$
$\mathcal{A}_2$	$\langle(0.3, 0.3)\rangle$	$\langle(0.8, 0.6)\rangle$	$\langle(0.3, 0.8)\rangle$	$\langle(0.4, 0.1)\rangle$
$\mathcal{A}_3$	$\langle(0.5, 0.3)\rangle$	$\langle(0.3, 0.2)\rangle$	$\langle(0.1, 0.9)\rangle$	$\langle(0.7, 0.7)\rangle$
$\mathcal{A}_4$	$\langle(0.3, 0.3)\rangle$	$\langle(0.5, 0.5)\rangle$	$\langle(0.3, 0.9)\rangle$	$\langle(0.7, 0.8)\rangle$
$\mathcal{A}_5$	$\langle(0.9, 0.8)\rangle$	$\langle(0.2, 0.5)\rangle$	$\langle(0.8, 0.7)\rangle$	$\langle(0.4, 0.8)\rangle$
$\mathcal{A}_6$	$\langle(0.7, 0.8)\rangle$	$\langle(0.6, 0.7)\rangle$	$\langle(0.1, 0.2)\rangle$	$\langle(0.2, 0.1)\rangle$

	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\mathcal{A}_1$	$\langle(0.1, 0.2)\rangle$	$\langle(0.9, 0.1)\rangle$	$\langle(0.3, 0.1)\rangle$	$\langle(0.4, 0.6)\rangle$
$\mathcal{A}_2$	$\langle(0.3, 0.1)\rangle$	$\langle(0.2, 0.5)\rangle$	$\langle(0.5, 0.6)\rangle$	$\langle(0.7, 0.4)\rangle$
$\mathcal{A}_3$	$\langle(0.1, 0.4)\rangle$	$\langle(0.1, 0.8)\rangle$	$\langle(0.5, 0.5)\rangle$	$\langle(0.8, 0.8)\rangle$
$\mathcal{A}_4$	$\langle(0.7, 0.2)\rangle$	$\langle(0.9, 0.7)\rangle$	$\langle(0.9, 0.6)\rangle$	$\langle(0.3, 0.3)\rangle$
$\mathcal{A}_5$	$\langle(0.8, 0.4)\rangle$	$\langle(0.6, 0.4)\rangle$	$\langle(0.5, 0.3)\rangle$	$\langle(0.8, 0.3)\rangle$
$\mathcal{A}_6$	$\langle(0.9, 0.6)\rangle$	$\langle(0.7, 0.3)\rangle$	$\langle(0.2, 0.3)\rangle$	$\langle(0.1, 0.4)\rangle$

Updating CFZNWA operator given in Eq. (16) for complex fuzzy sets as

$$CFWA (z_1, z_2, \ldots, z_n) = \left(1 - \prod_{i=1}^{n} (1 - c_i)^{\alpha_i}, \ 1 - \prod_{i=1}^{n} \left(1 - \omega_{c_i} \right)^{\alpha_i} \right)$$

And using score function in Eq. (27), we have $S(R_1) = 0.4490$, $S(R_2) = 0.4684$, $S(R_3) = 0.9367$, $S(R_4) = 0.5992$, $S(R_5) = 0.5467$ and $S(R_6) = 0.5097$ and the ranking is $\mathcal{A}_4 > \mathcal{A}_5 > \mathcal{A}_6 > \mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_2$. From Table 3, although the best alternative stays same, the second alternative falls into the last place. This change shows how important reliability is when making decisions.

Figures 2 and 3 depicts the ranking change of alternatives as the method changes. The ranking is almost same for the aggregation operators of proposed method. It indicates the reliability of the proposed method. However, when the data environment change, (reduced sets in these cases) rankings also change. When problem data is updated for Z-numbers that is explained in item II, the alternative $\mathcal{A}_4$ jumps to first place while $\mathcal{A}_1$ drops to fifth place. This is an important change and is due to the simplification of the proposed problem values, which means loss of information.

Additionally, if the restriction and reliability operations of Z-numbers are taken the same as in the sum example below:

$$z_1 \oplus z_2 = \left\langle \left(1 - \prod_{i=1}^{2} (1 - c_i), \ 1 - \prod_{i=1}^{2} \left(1 - \omega_{c_i} \right) \right), \right.$$
$$\left. \left(1 - \prod_{i=1}^{2} (1 - r_i), \ 1 - \prod_{i=1}^{2} \left(1 - \omega_{r_i} \right) \right) \right\rangle$$

rankings also change when compared to proposed operators. But the point to be considered here is that it does not show similarity with the results found with the Z-number. Rankings completely alters except the worst alternative. There is no doubt that taking same operations for both parts of Z-numbers affect the results intensively.

Moreover, if the second part of CFZNSs taken as empty set, the proposed approach reduces directly to CFSs since the restriction part consists of complex data and fuzzy numbers. When rankings are compared between the proposed method and the CFNs, the effect of reliability can be seen clearly. Best alternatives stay same but the alternative $\mathcal{A}_2$ drops to last place. The reason of $\mathcal{A}_2$ is the second alternative in our proposed approach is the reliability values given in Table 3. These values are relatively higher than other reliability values which can be interpreted as the impact of reliability.

Numerical application with a real dataset

In the previous section, we demonstrated the power of the proposed method when applied to data expressed with CFZNs. First, we utilized hypothetical data and applied the CFZNWA and CFZNWG operators to aggregate CFZNs, obtaining the results. Next, we validated the proposed method using three test criteria, thereby measuring its consistency and validity. Finally, we compared the results with those obtained from other sets and observed changes in the rankings. Now, to see the real effect of the proposed method, we use a real example from UC Irvine Machine Learning Repository [3] that can be expressed as CFZNs.

The Iris flower dataset is a foundational benchmark in the field of classification, holding a prominent place in the literature on machine learning and statistics. This widely used dataset consists of 150 samples, equally distributed across three distinct Iris species: Iris-setosa, Iris-versicolor and Iris-virginica. Each sample is characterized by four features representing petal and sepal dimensions in centimetres (cm):

c_1 : sepal length, c_2 : sepal width, c_3 : petal length and c_4 : petal width.

The first three instances of each Iris species expressed as S_1, S_2 and S_3 are provided in Table 4.

Notably, two of the three classes are linearly separable, while the remaining class exhibits partial overlap with the others in feature space. This characteristic makes the Iris dataset a valuable tool for evaluating and comparing the performance of various classification algorithms. In contrast to previous approaches, we express the data as CFZNs to facilitate a decision-making problem. Sepal length and width are considered as the restriction part of the first criteria, while petal length and width are taken as the reliability part of the second criteria. Since these datasets are real samples, the reliability of the inputs is maximum and therefore, the reliability component for both criteria are set to 1. To represent the data as CFZNs, we transform the existing values into four-element vectors for each criterion, as given in Table 5.

To apply the values given in Table 5 to CFZNs, we first need to normalize these values using the one of the normalization steps given below, which is the most commonly used in the literature:

$$N_{ij} = \frac{a_{ij}}{\sqrt{\sum_{i=1}^{m} a_{ij}^2}} (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n) \tag{28}$$

where m is sample numbers, n is criteria and a_{ij} is each element of each criterion. The we obtain the CFZN decision matrix as provided in Table 6.

Applying the CFZNWA operator given in Eq. (25), the aggregated values R_i are obtained as

$$R_1 = \langle (0.0493, 0.0529), (0.0816, 0.0816) \rangle$$
$$R_2 = \langle (0.0479, 0.0461), (0.0816, 0.0816) \rangle$$
$$R_3 = \langle (0.0455, 0.0488), (0.0816, 0.0816) \rangle$$

Table 5 Iris data expressed as four elements

	C_1	C_2
Iris-setosa		
S_1	$\langle(5.1, 3.5), (1, 1)\rangle$	$\langle(1.4, 0.2), (1, 1)\rangle$
S_2	$\langle(4.9, 3.0), (1, 1)\rangle$	$\langle(1.4, 0.2), (1, 1)\rangle$
S_3	$\langle(4.7, 3.2), (1, 1)\rangle$	$\langle(1.3, 0.2), (1, 1)\rangle$
$\vdots$	$\vdots$	$\vdots$
Iris-versicolor		
S_{51}	$\langle(7.0, 3.2), (1, 1)\rangle$	$\langle(4.7, 1.4), (1, 1)\rangle$
S_{52}	$\langle(6.4, 3.2), (1, 1)\rangle$	$\langle(4.5, 1.5), (1, 1)\rangle$
S_{53}	$\langle(6.9, 3.1), (1, 1)\rangle$	$\langle(4.9, 1.5), (1, 1)\rangle$
$\vdots$	$\vdots$	$\vdots$
Iris-virginica		
S_{101}	$\langle(6.3, 3.3), (1, 1)\rangle$	$\langle(6.0, 2.5), (1, 1)\rangle$
S_{102}	$\langle(5.8, 2.7), (1, 1)\rangle$	$\langle(5.1, 1.9), (1, 1)\rangle$
S_{103}	$\langle(7.1, 3.0), (1, 1)\rangle$	$\langle(5.9, 2.1), (1, 1)\rangle$
$\vdots$	$\vdots$	$\vdots$

$\vdots$

$R_{51} = \langle(0.0947, 0.0826), (0.0816, 0.0816)\rangle$

$R_{52} = \langle(0.0885, 0.0855), (0.0816, 0.0816)\rangle$

$R_{53} = \langle(0.0959, 0.0842), (0.0816, 0.0816)\rangle$

$\vdots$

$R_{101} = \langle(0.1027, 0.1160), (0.0816, 0.0816)\rangle$

$R_{102} = \langle(0.0904, 0.0906), (0.0816, 0.0816)\rangle$

$R_{103} = \langle(0.1072, 0.1003), (0.0816, 0.0816)\rangle$

$\vdots$

Using the score function given in Eq. (27) we have $S(R_1) = 0.0042, S(R_2) = 0.0038, S(R_3) = 0.0039$, $\dots$, $S(R_{51}) = 0.0072$, $S(R_{52}) = 0.0071$, $S(R_{53}) = 0.0074$, $\dots$, $S(R_{101}) = 0.0089$, $S(R_{102}) = 0.0074$, $S(R_{103}) = 0.0085$ and so on until $S(R_{150}) = 0.0075$.

Sorting the scores in ascending order demonstrates a clear separation between Iris species. The first 49 sample out of 50 belong to Iris-setosa class. Similarly, when we look rankings between 50 and 100, 45 out of 50 sample belong to Iris-versicolor and 46 out of 50 sample belong to Iris-virginica ranking between 100 and 150. Here, one sample that should belong to the Iris-setosa class is misclassified as Iris-versicolor. Similarly, of the Iris-versicolor class samples, one is misclassified as Iris-setosa and the remaining four is assigned to the Iris-virginica. Finally, all four samples that should belong to the Iris-virginica class is misclassified as Iris-versicolor. These observations demonstrate a 93.33% match, which is a promising result since the proposed method is not a classification or clustering method.

Table 6 CFZN decision matrix

	C_1		C_2
Iris-setosa			
S_1	$\langle(0.0706, 0.0927), (0.0816, 0.0816)\rangle$		$\langle(0.0275, 0.0115), (0.0816, 0.0816)\rangle$
S_2	$\langle(0.0678, 0.0794), (0.0816, 0.0816)\rangle$		$\langle(0.0275, 0.0115), (0.0816, 0.0816)\rangle$
S_3	$\langle(0.0650, 0.0847), (0.0816, 0.0816)\rangle$		$\langle(0.0256, 0.0115), (0.0816, 0.0816)\rangle$
$\vdots$	$\vdots$		$\vdots$
Iris-versicolor			
S_{51}	$\langle(0.0969, 0.0847), (0.0816, 0.0816)\rangle$		$\langle(0.0925, 0.0805), (0.0816, 0.0816)\rangle$
S_{52}	$\langle(0.0885, 0.0847), (0.0816, 0.0816)\rangle$		$\langle(0.0885, 0.0863), (0.0816, 0.0816)\rangle$
S_{53}	$\langle(0.0955, 0.0821), (0.0816, 0.0816)\rangle$		$\langle(0.0964, 0.0863), (0.0816, 0.0816)\rangle$
$\vdots$	$\vdots$		$\vdots$
Iris-virginica			
S_{101}	$\langle(0.0872, 0.0874), (0.0816, 0.0816)\rangle$		$\langle(0.1181, 0.1438), (0.0816, 0.0816)\rangle$
S_{102}	$\langle(0.0802, 0.0715), (0.0816, 0.0816)\rangle$		$\langle(0.1003, 0.1093), (0.0816, 0.0816)\rangle$
S_{103}	$\langle(0.0982, 0.0794), (0.0816, 0.0816)\rangle$		$\langle(0.1161, 0.1208), (0.0816, 0.0816)\rangle$
$\vdots$	v $\vdots$		$\vdots$

Table 7 Comparison of classification rate

Methods	Match rate
CFZNWG operator	93.33%
Z-numbers with ZNWA operator	92%
Z-numbers with same operations	89.33%
Complex fuzzy set	92%
Proposed approach	93.33%

Table 8 Comparison of rates with classification algorithms

Algorithm	Classification rate (%)
[30] PSO	84.80
[2] PSOcf	86.20
[32] TFBO	90.80
[41] Jaya	80.93
[33] GSA	0.00
[38] BBO	83.00
[42] CoDE	67.06
[37] FPSO	84.73
[22] PSONHM	**93.40**
Proposed method	**93.33**

Comparison and discussion

We have compared the proposed method in Sub-Sect. "Comparison and discussion" and have stated that there are few cases where we can compare the proposed CFZNS. This is because the proposed set and its operations are new, and we can only compare proposed set with the special cases of this set. After that, we compared the results with CFZNWG operator, Z-numbers, Z-numbers with same arithmetic operations and complex fuzzy sets and observed a change of ranking as given in Table 3. Using same idea, we can compare the results obtained by real data with these operations. In other words, the CFZN decision matrix given in Table 6 can be rewritten according to item I, II, III and VI in 4.3. Then, operating same calculations we obtain the results given in Table 7.

The results applied by CFZNWG operator give the same matching rate as the proposed method which shows the stability and the reliability of the proposed method. Additionally, the fact that the proposed set outperforms the other sets demonstrates its superiority. Furthermore, the similarity of the results obtained with other sets to those of the proposed set demonstrates the suitability of both the proposed set and its special cases for real-world data applications.

When we compare these results to the classification methods we notice a significant outcome. Li [22] solved same Iris flower classification [3] with particle swarm optimization based on neighbourhood historical memory (PSONHM) and compared results with other classification algorithms. Comparison of all classification rates with our proposed set given in Table 8.

The classification algorithms given in Table 8 are the proper tools for classify the data. However, while existing algorithms, excluding PSONHM and our proposed method, presents limitations in accurately classifying the real-world Iris data, both PSONHM and our method achieve almost same performance. This is noteworthy considering our method is not specifically designed for classification tasks. Additionally, it's important to acknowledge that all these algorithms require computational resources to converge to their final results.

Conclusion

In this paper, the concept of CFZNS is firstly introduced by extending the Z-number environment to complex environment to solve comprehensive uncertain information. First, fundamental operational laws are given, and their properties are investigated. Then, score and accuracy functions are defined to measure the four-dimensional data. Further, CFZNWA and CFZNWG averaging operators are introduced based on given fundamental arithmetic operations. Their desirable properties such as idempotency, monotonicity and boundedness are given and proved. Moreover, aggregation operators and score function are combined to develop an MCDM method, and an attractive numerical example is selected to show its advantages. Additionally, validation of the proposed operations is given, and a detailed comparative study is performed to demonstrate the effectiveness of the method. Moreover, a real-world data application from the UCI database is carried out and the results are compared with both special cases of the proposed set and the classification algorithms. Considering the studies in the literature, the proposed CFZNS has the following contributions:

(1) CFZNS can transmit the two-dimensional fuzzy information contained in the Z-number to four-dimensional information. The proposed CFZNS is both a generalization of the Z-numbers and the CFSs.
(2) This work presents the fundamental arithmetic operations of CFZNS, which can be taken as arithmetic operations for Z-numbers when ω_c and ω_r are taken as 0. besides, the given operations are also consistent with the Z-numbers. However, the numerical results show that the superiority of the CFZNS when compared to Z-numbers.
(3) Item IV in the comparison section proves the importance and effectiveness of the reliability in decision making problems.

(4) Considering that a decision-making problem can be expressed with all fuzzy sets in the literature, the proposed method brings a new perspective to the way decision making problems are expressed in the literature.

(5) Sect. "Numerical Application with a Real Dataset" confirms that real-word situations can be expressed as CFZNs and the results are promising.

Although the CFZNS is a novel approach to handle uncertain information, it has surely some limitations. Since z-numbers can be considered a new concept compared to all fuzzy sets, there is still more work to be done in this area. For example, the "processing with words" that Zadeh mentioned for the reliability part is not fully understood. Therefore, the operations proposed for reliability part can have some shortcomings. This lack of information affects the more generalized sets and one of them is our proposed work. Additionally, to use real data on this method is not possible yet since the lack of information about fuzzification on Z-numbers and complex sets. However, these limitations open a new field of research about application of real data. Although the proposed set is a generalization of complex fuzzy sets and Z-numbers, it is also a novel set approach. For this reason, it opens many new ranges of study, from aggregation operators to distance measures, from decision-making methods to entropy. Additionally, since decision-making in this proposed set give encouraging results, classification problems and clustering applications may be intriguing for future studies.

Data availability statement Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors report there are no competing interests to declare.

References

1. Beliakov G, Bustince H, Goswami DP, Mukherjee UK, Pal NR (2011) On averaging operators for Atanassov's intuitionistic fuzzy sets. Inf Sci (NY) 181:1116–1124. https://doi.org/10.1016/j.ins.2010.11.024

2. Clerc M, Kennedy J (2002) The particle swarm-explosion, stability, and convergence in a multidimensional complex space. IEEE Trans Evol Comput 6:58–73. https://doi.org/10.1109/4235.985692

3. Fisher RA (1988) Iris. UCI Machine Learning Repository. https://doi.org/10.24432/C56C76

4. Gai L, Liu H, Wang Y, Xing Y (2023) Green supplier selection and order allocation using linguistic Z-numbers MULTIMOORA method and bi-objective non-linear programming. Fuzzy Optim Decis Mak 22:267–288. https://doi.org/10.1007/s10700-022-09392-1

5. Garg H, Rani D (2020) Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their applications to decision-making process. Artif Intell Rev 53:3595–3620. https://doi.org/10.1007/s10462-019-09772-x

6. Gwak J, Garg H, Jan N (2023) Hybrid integrated decision-making algorithm for clustering analysis based on a bipolar complex fuzzy and soft sets. Alex Eng J 67:473–487. https://doi.org/10.1016/j.aej.2022.12.003

7. Haktanir E, Kahraman C (2022) New product design using Chebyshev's inequality based interval-valued intuitionistic Z-fuzzy QFD method. Informatica 33:1–33. https://doi.org/10.15388/22-INFOR476

8. Hu L (2023) A hybrid generalized TODIM based risk prioritization method for failure mode and effect analysis with linguistic Z-numbers. J Intell Fuzzy Syst 44:7935–7955. https://doi.org/10.3233/JIFS-223132

9. Jafar MN, Saeed M (2022) Matrix theory for neutrosophic hypersoft set and applications in multiattributive multicriteria decision-making problems J. Math 2022:6666408. https://doi.org/10.1155/2022/6666408

10. Jafar MN, Saeed M, Khan KM, Alamri FS, Khalifa HAE-W (2022) Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. IEEE Access 10:11220–11235. https://doi.org/10.1109/ACCESS.2022.3144306

11. Jafar MN, Saeed M, Saqlain M, Yang M-S (2021) Trigonometric similarity measures for neutrosophic hypersoft sets with application to renewable energy source selection. IEEE Access 9:129178–129187. https://doi.org/10.1109/ACCESS.2021.3112721

12. Jia Q, Hu J, Herrera-Viedma E (2022) A novel solution for Z-numbers based on complex fuzzy sets and its application in decision-making system. IEEE Trans Fuzzy Syst 30:4102–4114. https://doi.org/10.1109/TFUZZ.2021.3138649

13. Karabacak M (2023) Interval neutrosophic multi-criteria group decision-making based on Aczel–Alsina aggregation operators. Comput Appl Math 42:136. https://doi.org/10.1007/s40314-023-02236-7

14. Kaur G, Garg H (2023) A novel algorithm for autonomous parking vehicles using adjustable probabilistic neutrosophic hesitant fuzzy set features. Expert Syst Appl 226:120101. https://doi.org/10.1016/j.eswa.2023.120101

15. Klement EP, Mesiar R (2005) Logical, algebraic, analytic and probabilistic aspects of triangular norms. Elsevier, Amsterdam

16. Klir G, Yuan B (1995) Fuzzy sets and fuzzy logic. Prentice Hall, New Jersey

17. Köseoğlu A (2022) Intuitionistic multiplicative set approach for green supplier selection problem using Todim method. J Univers Math 5:149–158. https://doi.org/10.33773/jum.1134398

18. Köseoğlu A (2024) Generalized correlation coefficients of intuitionistic multiplicative sets and their applications to pattern recognition and clustering analysis. J Exp Theor Artif Intell. https://doi.org/10.1080/0952813X.2024.2323039

19. Köseoğlu A (2022) A comparative decision-making for electronic product purchases during a pandemic. Gümüşhane Üniversitesi Fen Bilim Enstitüsü Derg 109–118. https://doi.org/10.17714/gumusfenbil.1001904

20. Köseoğlu A, Şahin R (2023) Correlation coefficients of simplified neutrosophic multiplicative sets and their applications in clustering analysis. J Ambient Intell Humaniz Comput 14:3383–3404. https://doi.org/10.1007/s12652-021-03475-4

21. Köseoğlu A, Şahin R, Merdan M (2020) A simplified neutrosophic multiplicative set-based TODIM using water-filling algorithm for the determination of weights. Expert Syst 37:e12515. https://doi.org/10.1111/exsy.12515

22. Li W (2018) Improving particle swarm optimization based on neighborhood and historical memory for training multi-layer perceptron. Information 9:16. https://doi.org/10.3390/info9010016

23. Lotfi R, Hazrati H, Ali SS, Sharifmousavi SM, Khanbaba A, Amra M (2023) Antifragile, sustainable and agile healthcare waste chain network design by considering blockchain, resiliency, robustness and risk. Cent Eur J Oper Res. https://doi.org/10.1007/s10100-023-00874-0

24. Lotfi R, Kargar B, Rajabzadeh M, Hesabi F, Özceylan E (2022) Hybrid fuzzy and data-driven robust optimization for resilience and sustainable health care supply chain with vendor-managed inventory approach. Int J Fuzzy Syst 24:1216–1231. https://doi.org/10.1007/s40815-021-01209-4

25. Lotfi R, Mehrjardi MS, MohajerAnsari P, Zolfaqari F, Afshar M (2023) Antifragile, sustainable, and agile supply chain network design by considering resiliency, robustness, risk, and environmental requirements. Environ Sci Pollut Res 30:106442–106459. https://doi.org/10.1007/s11356-023-29488-4

26. Lotfi R, Mohammad Shafiei R, Komeleh MG, Pasha FG, Ferasat M (2023) Vaccine supply chain network design by considering viability, robustness and risk. J Eng Res. https://doi.org/10.1016/j.jer.2023.10.007

27. Mahmood T, ur Rehman U (2022) A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures. Int J Intell Syst 37:535–567. https://doi.org/10.1002/int.22639

28. Mahmood T, ur Rehman U, Ali Z, (2023) Analysis and application of Aczel-Alsina aggregation operators based on bipolar complex fuzzy information in multiple attribute decision making. Inf Sci (NY) 619:817–833. https://doi.org/10.1016/j.ins.2022.11.067

29. Mao J, Meng X, Ding D (2022) Fuzzy set-membership filtering for discrete-time nonlinear systems. IEEE/CAA J Autom Sin 9:1026–1036. https://doi.org/10.1109/JAS.2022.105416

30. Mendes R, Cortez P, Rocha M, Neves J (2002) Particle swarms for feedforward neural network training. In: Proceedings of the international joint conference on neural networks, pp 1895–1899

31. Ramot D, Milo R, Friedman M, Kandel A (2002) Complex fuzzy sets. IEEE Trans Fuzzy Syst 10:171–186. https://doi.org/10.1109/91.995119

32. Rao RV, Savsani VJ, Vakharia DP (2011) Teaching-learning-based optimization: a novel method for constrained mechanical design optimization problems. CAD Comput Aided Des 43:303–315. https://doi.org/10.1016/j.cad.2010.12.015

33. Rashedi E, Nezamabadi-pour H, Saryazdi S (2009) GSA: a gravitational search algorithm. Inf Sci (NY) 179:2232–2248. https://doi.org/10.1016/j.ins.2009.03.004

34. Şahin R, Altun F (2020) Decision making with MABAC method under probabilistic single-valued neutrosophic hesitant fuzzy environment. J Ambient Intell Humaniz Comput 11:4195–4212. https://doi.org/10.1007/s12652-020-01699-4

35. Sari IU, Kahraman C (2021) Intuitionistic Fuzzy Z-numbers BT—intelligent and fuzzy techniques: smart and innovative solutions. In: Kahraman C, Cevik Onar S, Oztaysi B, Sari IU, Cebi S, Tolga AC (eds). Springer International Publishing, Cham, pp 1316–1324

36. Seikh MR, Mandal U (2022) Multiple attribute group decision making based on quasirung orthopair fuzzy sets: application to electric vehicle charging station site selection problem. Eng Appl Artif Intell 115:105299. https://doi.org/10.1016/j.engappai.2022.105299

37. Shi Y, Eberhart RC (2001) Fuzzy adaptive particle swarm optimization. In: Proceedings of the IEEE conference on evolutionary computation, ICEC. IEEE, pp 101–106

38. Simon D (2008) Biogeography-based optimization. IEEE Trans Evol Comput 12:702–713. https://doi.org/10.1109/TEVC.2008.919004

39. Triantaphyllou E (2000) Multi-criteria decision making methods BT—multi-criteria decision making methods: a comparative study. In: Triantaphyllou E (ed). Springer US, Boston, pp 5–21

40. Ünver M, Olgun M, Türkarslan E (2022) Cosine and cotangent similarity measures based on Choquet integral for spherical fuzzy sets and applications to pattern recognition. J Comput Cogn Eng 1:21–31. https://doi.org/10.47852/bonviewJCCE2022010105

41. Venkata Rao R (2016) Jaya: a simple and new optimization algorithm for solving constrained and unconstrained optimization problems. Int J Ind Eng Comput 7:19–34. https://doi.org/10.5267/J.IJIEC.2015.8.004

42. Wang Y, Cai Z, Zhang Q (2011) Differential evolution with composite trial vector generation strategies and control parameters. IEEE Trans Evol Comput 15:55–66. https://doi.org/10.1109/TEVC.2010.2087271

43. Ye J, Du S, Yong R (2023) Hybrid weighted arithmetic and geometric averaging operator of cubic Z-numbers and its decision-making method. J Control Decis. https://doi.org/10.1080/23307706.2023.2181235

44. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

45. Zadeh LA (2011) A note on Z-numbers. Inf Sci (NY) 181:2923–2932. https://doi.org/10.1016/j.ins.2011.02.022

46. Zheng W, Zhang Z, Sun F, Wen S (2022) Robust stability analysis and feedback control for networked control systems with additive uncertainties and signal communication delay via matrices transformation information method. Inf Sci (NY) 582:258–286. https://doi.org/10.1016/j.ins.2021.09.005